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Involutive automorphisms of the affine Kac–Moody algebras $A_l^{(1)}$ for $l \geqslant 3$ by the matrix method

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Abstract. The determination of all the conjugacy classes of involutive automorphisms of the affine Kac–Moody algebras $A_l^{(1)}$ for $l \geqslant 3$ is presented using the matrix formulation of automorphisms of an affine Kac–Moody algebra that was developed in a previous paper.

1. Introduction

In a previous paper [1] a general matrix formulation of the automorphisms of untwisted affine Kac–Moody algebras was developed, and was subsequently applied to the special cases of $A_1^{(1)}$ [2] and $A_2^{(1)}$ [3]. (These papers will henceforth be referred to as papers I, II and III respectively.) However, as was noted in papers II and III, both $A_1^{(1)}$ and $A_2^{(1)}$ have some special features that are absent in the general case of $A_l^{(1)}$ with $l \geqslant 3$, and these special features help to simplify the analysis. The present paper is devoted to finding the conjugacy classes of the involutive automorphisms of the more complicated algebras $A_l^{(1)}$ with $l \geqslant 3$.

The notations and conventions that will be employed in the present paper are those defined in paper I (with the additional convention here that equation labels such as (6) and (I.6) refer to the sixth numbered equation of the present paper and of paper I respectively). When the untwisted affine Kac–Moody algebra $\tilde{\mathcal{L}}$ is $A_l^{(1)}$, the corresponding simple Lie algebra is A_l, for which has rank l. The generalized Cartan matrix of $A_l^{(1)}$ for $l \geqslant 3$ is

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{1}$$

The $l+1$ simple roots of $A_l^{(1)}$ are α_j for $j = 0, \ldots, l$, and as the highest root $\alpha_{\rm H}^0$ of A_l is $\sum_{j=1}^{l} \alpha_j^0$, the relation $\alpha_0 = \delta - \alpha_{\rm H}$ implies that the root δ is given by

$$\delta = \alpha_0 + \sum_{j=1}^{l} \alpha_j \tag{2}$$

and so

$$c = h_\delta = h_{\alpha_0} + \sum_{j=1}^{l} h_{\alpha_j}. \tag{3}$$

For these simple roots

$$\langle \alpha_j, \alpha_k \rangle = (1/\{2(l+1)\}) A_{jk} \tag{4}$$

for $j, k = 0, \ldots, l$.

Let $\boldsymbol{\Gamma}$ be the $(l+1)$-dimensional irreducible representation of A_l in which

$$\boldsymbol{\Gamma}(h^0_{\alpha^0_j}) = \mathbf{h}^0_{\alpha^0_j} = (1/\{2(l+1)\})\,(\mathbf{e}_{j,j} - \mathbf{e}_{j+1,j+1}) \tag{5}$$

for $j = 1, \ldots, l$, and

$$\boldsymbol{\Gamma}(e^0_{\alpha^0}) = \mathbf{e}^0_{\alpha^0} = (1/\{2(l+1)\})^{1/2} \mathbf{e}_{p,q} \tag{6}$$

$$\boldsymbol{\Gamma}(e^0_{-\alpha^0}) = \mathbf{e}^0_{-\alpha^0} = -(1/\{2(l+1)\})^{1/2} \mathbf{e}_{q,p} \tag{7}$$

where $\alpha^0 = \sum_{j=p}^{q-1} \alpha^0_j$ (for $p < q$) and where $\mathbf{e}_{j,k}$ is the $(l+1) \times (l+1)$ matrix for which $(\mathbf{e}_{j,k})_{rs} = \delta_{jr}\delta_{ks}$ for $j, k, r, s = 1, \ldots, l$. The value of the Dynkin index of this representation is given by $\gamma = 1/\{2(l+1)\}$.

As in the case of $A_2^{(1)}$, this representation of $A_l^{(1)}$ for $l \geqslant 3$ is *not* equivalent to its contragredient representation. Thus for $A_l^{(1)}$ with $l \geqslant 3$ the set of type 1b involutive automorphisms does not coincide with the set of type 1a involutive automorphisms and the set of type 2b involutive automorphisms does not coincide with the set of type 2a involutive automorphisms. The type 1a and type 1b sets both divide into two disjoint subsets with $u = 1$ and $u = -1$, whereas for the determination of the representatives of the conjugacy classes of the type 2a and the type 2b sets it is sufficient to let $u = 1$. Consequently there are six sets of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ to be considered. These are:

(i) type 1a involutive automorphisms with $u = 1$, which will be analysed in section 2;

(ii) type 1a involutive automorphisms with $u = -1$, which will be analysed in section 3;

(iii) type 1b involutive automorphisms with $u = 1$, which will be analysed in section 4;

(iv) type 1b involutive automorphisms with $u = -1$, which will be analysed in section 5;

(v) type 2a involutive automorphisms (with $u = 1$), which will be analysed in section 6;

(vi) type 2b involutive automorphisms (with $u = 1$), which will be analysed in section 7.

The conclusions concerning involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ are summarized in section 8, and compared there with previous related work.

As was demonstrated in papers II and III, one can rely completely on the matrix formulation of automorphisms, the only structural ingredient needed being some knowledge of the root transformations of the corresponding simple Lie algebra A_l. As mentioned in section 3 of paper I, every conjugacy class of involutive automorphisms of $\tilde{\mathcal{L}}$ contains at least one Cartan-preserving involutive automorphism, and each such Cartan-preserving involutive automorphism is associated with an involutive root transformation τ^0 of $\tilde{\mathcal{L}}$.

For $l \geqslant 2$ the important basic root-preserving transformations of $A_l^{(1)}$ are:
(i) $\tau^0 = E$ (the identity), for which

$$\tau^0(\alpha_j^0) = \alpha_j^0 \qquad (j = 1, \ldots, l) \tag{8}$$

(ii) $\tau^0 = S^0_{\alpha_1^0}$ (the primitive Weyl transformation associated with the simple root α_1^0), for which

$$\begin{aligned} \tau^0(\alpha_1^0) &= -\alpha_1^0 \\ \tau^0(\alpha_2^0) &= \alpha_1^0 + \alpha_2^0 \\ \tau^0(\alpha_j^0) &= \alpha_j^0 \qquad (j = 3, \ldots, l) \end{aligned} \tag{9}$$

(iii) $\tau^0 = S^0_{\alpha_j^0}$ (the primitive Weyl transformation associated with the simple root α_j^0) (with $3 \leqslant j \leqslant l-1$), for which

$$\begin{aligned} \tau^0(\alpha_k^0) &= \alpha_k^0 \qquad (k = 1, \ldots, j-2) \\ \tau^0(\alpha_{j-1}^0) &= \alpha_{j-1}^0 + \alpha_j^0 \\ \tau^0(\alpha_j^0) &= -\alpha_j^0 \\ \tau^0(\alpha_{j+1}^0) &= \alpha_{j+1}^0 + \alpha_j^0 \\ \tau^0(\alpha_k^0) &= \alpha_k^0 \qquad (k = j+2, \ldots, l). \end{aligned} \tag{10}$$

(iv) $\tau^0 = S^0_{\alpha_l^0}$ (the primitive Weyl transformation associated with the simple root α_l^0), for which

$$\begin{aligned} \tau^0(\alpha_j^0) &= \alpha_j^0, (j = 1, \ldots, l-2) \\ \tau^0(\alpha_{l-1}^0) &= \alpha_{l-1}^0 + \alpha_l^0 \\ \tau^0(\alpha_l^0) &= -\alpha_l^0 \end{aligned} \tag{11}$$

(v) $\tau^0 = \rho^0$ (the Dynkin diagram symmetry operation for A_l), for which

$$\tau^0(\alpha_j^0) = \alpha_{l+1-j}^0 \qquad (j = 1, \ldots, l) \tag{12}$$

(vi) $\tau^0 = \tau^0_{\text{Cartan}}$ (the Cartan involution for A_l), for which

$$\tau^0(\alpha_j^0) = -\alpha_j^0 \qquad (j = 1, \ldots, l). \tag{13}$$

For $l \geqslant 2$ the group of root-preserving transformations of A_l is isomorphic to the direct product of the subgroup of order 2 $\{E, \tau^0_{\text{Cartan}}\}$ with the subgroup of Weyl transformations of A_l. Consequently the group of root-preserving transformations of A_l has order $2\{(l+1)!\}$ (for $l \geqslant 2$). Although neither the Cartan involution τ^0_{Cartan} nor the Dynkin diagram symmetry operation ρ^0 can be expressed as a Weyl transformation, the product $\tau^0_{\text{Cartan}} \circ \rho^0$ does coincide with a Weyl transformation. This implies that for $l \geqslant 2$ ρ^0 can always be written in the form $\rho^0 = S \circ \tau^0_{\text{Cartan}}$ for some Weyl transformation S.

For $l \geqslant 2$ the Weyl group of A_l has $[l]$ conjugacy classes of *involutive* transformations, where $[a]$ denotes the largest integer not exceeding a. These consist of the identity E, together with $[\frac{1}{2}(l+1)]$ conjugacy classes of order two Weyl transformations, for which a set of representatives may be taken to be $\{S^0_{\alpha^0_1}\}$ for A_2, $\{S^0_{\alpha^0_1}, S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3}\}$ for A_3 and A_4, $\{S^0_{\alpha^0_1}, S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3}, S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ S^0_{\alpha^0_5}\}$ for A_5 and A_6, and so on. (It should be noted the the primitive Weyl reflections in each of these sets mutually commute.) Consequently the group of root-preserving transformations of A_l for $l \geqslant 2$ has $2[l]$ conjugacy classes of *involutive* transformations, for which the representatives may be taken to be the $[l]$ Weyl group elements just listed, together with their products with τ^0_{Cartan}. The last of these conjugacy classes, that is the one containing $S^0_{\alpha^0_1} \circ \ldots \circ S^0_{\alpha^0_k} \circ \tau^0_{\text{Cartan}}$ with $k = 2[\frac{1}{2}(l+1)] - 1$, also contains ρ^0, and it is actually more convenient to take ρ^0 as its representative.

2. Study of the involutive automorphisms of $A^{(1)}_l$ for $l \geqslant 3$ of type 1a with $u = 1$

2.1. Relevant conjugacy classes for the involutive automorphisms of $A^{(1)}_l$ for $l \geqslant 3$ of type 1a with $u = 1$

It is easily shown that there exists no non-singular $(l+1) \times (l+1)$ matrix $\mathbf{U}(t)$ that satisfies the conditions:

$$\mathbf{U}(t)\mathbf{h}^0_{\alpha^0_j}\mathbf{U}(t)^{-1} = -\mathbf{h}^0_{\alpha^0_j} \tag{14}$$

for $j = 1, \ldots, l$. Consequently there are *no* type 1a involutive automorphisms of $A^{(1)}_l$ for $l \geqslant 3$ with $u = 1$ corresponding to the Cartan involution τ^0_{Cartan} of (13). Similarly there are *no* type 1a involutive automorphisms of $A^{(1)}_l$ for $l \geqslant 3$ with $u = 1$ corresponding to the root transformations that are products of Weyl transformations and τ^0_{Cartan}, so attention may be concentrated on those associated with the Weyl transformations alone.

It is convenient to divide the conjugacy classes of Weyl transformations into the following three sets:

(A) the conjugacy class consisting of E alone;

(B) the conjugacy classes with representatives: $S^0_{\alpha^0_1}$, $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3}$, $\ldots$, $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \ldots \circ S^0_{\alpha^0_k}$ with

$$k = \begin{cases} l-2 & \text{if } l \text{ is odd} \\ l-1 & \text{if } l \text{ is even} \end{cases} \tag{15}$$

(C) if l is *odd*, the conjugacy class with representative $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \ldots \circ S^0_{\alpha^0_l}$.

The type 1a involutive automorphisms with $u = 1$ corresponding to these sets will now be considered in turn.

2.2. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = 1$ corresponding to set (A)

The most general $(l+1) \times (l+1)$ matrix $\mathbf{U}(t)$ that satisfies

$$\mathbf{U}(t)\mathbf{h}^0_{\alpha^0_j}\mathbf{U}(t)^{-1} = \mathbf{h}^0_{\alpha^0_j} \tag{16}$$

for $j = 1, \dots, l$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by

$$\mathbf{U}(t) = \begin{pmatrix} \eta'_1 t^{k'_1} & 0 & 0 & \cdots & 0 \\ 0 & \eta'_2 t^{k'_2} & 0 & \cdots & 0 \\ 0 & 0 & \eta'_3 t^{k'_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \eta'_{l+1} t^{k'_{l+1}} \end{pmatrix}$$

where, for $j = 1, \dots, l+1$, η'_j are arbitrary non-zero complex numbers and k'_j are arbitrary integers. However, (I.111) shows that $(\eta'_1)^{-1} t^{-k'_1}\mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_k = (\eta'_k)/(\eta'_1)$ and $k_k = k'_k - k'_1$ for $k = 2, \dots, l+1$, it follows that the most general automorphism of type 1a with $u = 1$ associated with the identity root transformation corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \eta_2 t^{k_2} & 0 & \cdots & 0 \\ 0 & 0 & \eta_3 t^{k_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \eta_{l+1} t^{k_{l+1}} \end{pmatrix} \tag{17}$$

where, for $j = 2, \dots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.136) now reduces to $\mathbf{U}(t)^2 = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $k_j = 0$ and $\eta_j = \pm 1$ for $j = 2, \dots, l+1$. That is

$$\mathbf{U}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \eta_2 & 0 & \cdots & 0 \\ 0 & 0 & \eta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \eta_{l+1} \end{pmatrix} \tag{18}$$

where $\eta_j = \pm 1$ for $j = 2, \dots, l+1$. The conditions (I.158), (I.160), (I.162) and (I.164) for the conjugacy of two type 1a involutive automorphisms corresponding to two matrices $\mathbf{U}_1(t)$ and $\mathbf{U}_2(t)$ of the form (18) (and to $u = 1$) all reduce to

$$\eta\mathbf{U}_1(t) = \mathbf{S}\mathbf{U}_2(t)\mathbf{S}^{-1} \tag{19}$$

where $\mathbf{S}$ is clearly independent of t and η is an arbitrary non-zero complex number. A necessary condition for (19) to be satisfied is that

$$\{\mathrm{tr}\mathbf{U}_1(t)\}^{l+1}/\{\det \mathbf{U}_1(t)\} = \{\mathrm{tr}\mathbf{U}_2(t)\}^{l+1}/\{\det \mathbf{U}_2(t)\}. \tag{20}$$

Suppose that the set $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ contains n_+ entries with value +1 and the remaining $l - n_+$ entries have value -1. The case $n_+ = l$ corresponds to $\mathbf{U}_1(t) = \mathbf{1}$ and hence to the identity automorphism, which, as always, is in a class of its own, and which can be disregarded in the analysis that follows. Of the rest, there are only $[\frac{1}{2}(l+1)]$ distinct values of $\{\mathrm{tr}\mathbf{U}(t)\}^{l+1}/\{\det \mathbf{U}(t)\}$, which may be taken to correspond to

$$n_+ = [\tfrac{1}{2}l], [\tfrac{1}{2}l] + 1, \ldots, l-1. \tag{21}$$

(Here again $[a]$ denotes the largest integer not exceeding a). It is then clear that (19) is also a sufficient condition for conjugacy. For each value of n_+ satisfying (21) the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ may be chosen so that

$$\begin{aligned} &\eta_1 = \eta_2 = \ldots = \eta_{n_+} = 1 \\ &\eta_{(n_++1)} = \eta_{(n_++2)} = \ldots = \eta_l = -1 \\ &\eta_{(l+1)} = 1. \end{aligned} \tag{22}$$

By (I.67), (I.71), (I.73)) and (I.138) this corresponds to the involutive automorphism

$$\begin{aligned} &\psi(h_{\alpha_j}) = h_{\alpha_j} \qquad (j = 0, \ldots, l) \\ &\psi(c) = c \\ &\psi(d) = d \\ &\psi(e_{\pm\alpha_j}) = -e_{\pm\alpha_j} \qquad (j = n_+ \text{ or } l) \\ &\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (\text{for all other} j). \end{aligned} \tag{23}$$

2.3. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = 1$ corresponding to set (B)*

It will now be shown that all the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = 1$ corresponding to set (B) are actually conjugate to those corresponding to the set (A). To see this it will be sufficient to study an example that incorporates all the general features. Consider therefore the case $l = 4$ with conjugacy class representative $S^0_{\alpha_1^0}$. The most general 5×5 matrix $\mathbf{U}(t)$ that satisfies

$$\begin{aligned} &\mathbf{U}(t)\mathbf{h}^0_{\alpha_1^0}\mathbf{U}(t)^{-1} = -\mathbf{h}^0_{\alpha_1^0} \\ &\mathbf{U}(t)\mathbf{h}^0_{\alpha_2^0}\mathbf{U}(t)^{-1} = \mathbf{h}^0_{\alpha_1^0} + \mathbf{h}^0_{\alpha_2^0} \\ &\mathbf{U}(t)\mathbf{h}^0_{\alpha_3^0}\mathbf{U}(t)^{-1} = \mathbf{h}^0_{\alpha_3^0} \\ &\mathbf{U}(t)\mathbf{h}^0_{\alpha_4^0}\mathbf{U}(t)^{-1} = \mathbf{h}^0_{\alpha_4^0} \end{aligned} \tag{24}$$

with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by

$$\mathbf{U}(t) = \begin{pmatrix} 0 & \eta_1' t^{k_1'} & 0 & 0 & 0 \\ \eta_2' t^{k_2'} & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_3' t^{k_3'} & 0 & 0 \\ 0 & 0 & 0 & \eta_4' t^{k_4'} & 0 \\ 0 & 0 & 0 & 0 & \eta_5' t^{k_5'} \end{pmatrix} \tag{25}$$

where, for $j = 1, \ldots, 5$, η'_j are arbitrary non-zero complex numbers and k'_j are arbitrary integers. However, (I.111) shows that $(\eta'_5)^{-1}t^{-k'_5}\mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_k = (\eta'_k)/(\eta'_5)$ and $k_k = k'_k - k'_5$ for $k = 2, \ldots, 5$, it follows that the most general automorphism of type 1a with $u = 1$ associated with this root transformation corresponds to

$$\mathbf{U}(t) = \begin{pmatrix} 0 & \eta_1 t^{k_1} & 0 & 0 & 0 \\ \eta_2 t^{k_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_3 t^{k_3} & 0 & 0 \\ 0 & 0 & 0 & \eta_4 t^{k_4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

where, for $j = 1, \ldots, 4$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.136) again reduces to $\mathbf{U}(t)^2 = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies

$$k_2 = -k_1 \qquad k_3 = 0 \qquad k_4 = 0 \qquad \eta_2 = (\eta_1)^{-1} \qquad \eta_3 = \pm 1 \qquad \eta_4 = \pm 1. \tag{27}$$

As $\mathbf{U}(t)$ is therefore diagonalizable with eigenvalues ± 1, so that the diagonalized form is the same as that of (18), and as the matrix $\mathbf{S}(t)$ of the associated similarity transformation and its inverse both have entries that are all Laurent polynomials in t, it follows that every type 1a involutive automorphism (with $u = 1$) associated with such a $\mathbf{U}(t)$ is conjugate to one of the type 1a involutive automorphisms (with $u = 1$) of the previous subsection.

On repeating this analysis for any conjugacy class representative of the set (B), one finds that each factor $S^0_{\alpha^0_j}$ that appears is associated with a 2×2 submatrix of $\mathbf{U}(t)$ of the form

$$\begin{pmatrix} 0 & \eta_j t^{k_j} \\ (\eta_j)^{-1} t^{-k_j} & 0 \end{pmatrix}$$

which appears in the (j, j), $(j, j+1)$, $(j+1, j)$ and $(j+1, j+1)$ positions of $\mathbf{U}(t)$ (as in (26)). Similarly, each factor $S^0_{\alpha^0_j}$ that is missing from the product $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \ldots$ is associated with a 2×2 diagonal submatrix of $\mathbf{U}(t)$ (in the (j, j), $(j, j+1)$, $(j+1, j)$ and $(j+1, j+1)$ positions) with diagonal entries ± 1. Moreover, with the assumptions made about the set (B), the $(l+1, l+1)$ element of $\mathbf{U}(t)$ *always* has the value 1. Then $\mathbf{U}(t)$ is always diagonalizable with eigenvalues ± 1, so that the diagonalized form is the same as that of (18), and that the matrix $\mathbf{S}(t)$ of the associated similarity transformation and its inverse both have entries that are all Laurent polynomials in t. Consequently *every* type 1a involutive automorphisms (with $u = 1$) associated with a root transformation of the set (B) is conjugate to one of the type 1a involutive automorphisms (with $u = 1$) of the set (A).

2.4. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = 1$ corresponding to set (C)*

With the assumption that l is *odd*, the most general $(l+1) \times (l+1)$ matrix $\mathbf{U}(t)$ that corresponds to $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \ldots \circ S^0_{\alpha^0_l}$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by

$$\mathbf{U}(t)=\begin{pmatrix} 0 & \eta'_1 t^{k'_1} & 0 & 0 & \dots & 0 & 0 \\ \eta'_2 t^{k'_2} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \eta'_3 t^{k'_3} & \dots & 0 & 0 \\ 0 & 0 & \eta'_4 t^{k'_4} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \eta'_l t^{k'_l} \\ 0 & 0 & 0 & 0 & \dots & \eta'_{l+1} t^{k'_{l+1}} & 0 \end{pmatrix}. \tag{28}$$

It is convenient to re-order the rows and columns of $\mathbf{U}(t)$ to give

$$\mathbf{U}(t)=\begin{pmatrix} 0 & 0 & \dots & 0 & \eta'_1 t^{k'_1} \\ 0 & 0 & \dots & \eta'_2 t^{k'_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \eta'_l t^{k'_l} & \dots & 0 & 0 \\ \eta'_{l+1} t^{k'_{l+1}} & 0 & \dots & 0 & 0 \end{pmatrix} \tag{29}$$

which actually corresponds to the root transformation $\rho^0 \circ \tau^0_{\text{Cartan}}$. As (I.111) shows that $(\eta'_1)^{-1} t^{-k'_1}\mathbf{U}(t)$ and $\mathbf{U}(t)$ both give the *same* automorphism, so on putting $\eta_k = (\eta'_k)/(\eta'_1)$ and $k_k = k'_k - k'_1$ for $k = 2, \dots, l+1$, it follows that the most general automorphism of type 1a with $u = 1$ associated with this root transformation corresponds to

$$\mathbf{U}(t)=\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & \eta_2 t^{k_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \eta_l t^{k_l} & \dots & 0 & 0 \\ \eta_{l+1} t^{k_{l+1}} & 0 & \dots & 0 & 0 \end{pmatrix} \tag{30}$$

where, for $j = 2, \dots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.136) again reduces to $\mathbf{U}(t)^2 = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$\eta_{l+1} = \eta_2 \eta_l = \eta_3 \eta_{l-1} = \dots = \eta_{(l+1)/2}\eta_{(l+3)/2} \tag{31}$$

and

$$k_{l+1} = k_2 + k_l = k_3 + k_{l-1} = \dots = k_{(l+1)/2} + k_{(l+3)/2}. \tag{32}$$

Then $\eta_2, \eta_3, \dots, \eta_{(l+3)/2}$ can be taken to be arbitrary non-zero complex numbers and $k_2, k_3, \dots, k_{(l+3)/2}$ can be taken to be arbitrary integers. Suppose that the type 1a involutive automorphism corresponding to this (and to $u = 1$) is conjugate to that corresponding to $u = 1$ and a $\mathbf{U}(t)$ matrix of the form (18). Invoking (I.158), taking the determinants of both sides, and using (32) gives as a *necessary* condition for conjugacy that $k_{(l+1)/2} + k_{(l+3)/2}$ must be *even*. (The same conclusion also follows from using (I.160), (I.162) and (I.164) in place of (I.158), as the $\mathbf{U}(t)$ matrix of (18) is actually independent of t.) The two sub-cases will now be considered separately.

2.4.1. $k_{(l+1)/2} + k_{(l+3)/2}$ assumed even. It will now be shown that in this case *all* the type 1a involutive automorphisms associated with these $\mathbf{U}(t)$ matrices (and $u = 1$) are conjugate to that associated with $u = 1$ and

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{(l+1)/2} \end{pmatrix} \tag{33}$$

which is a special case of (18). The first step in this demonstration is to define $\mathbf{S}(t)$ to be the $(l+1) \times (l+1)$ diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} 1 & \text{for } j = (l+3)/2, \ldots, l+1 \\ (\eta_{l+2-j}/\eta_j)^{1/2} t^{(k_{l+2-j}-k_j)/2} & \text{for } j = 1, \ldots, (l+1)/2. \end{cases} \tag{34}$$

Then, with $\mathbf{U}(t)$ as defined in (30),

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \eta t^k \mathbf{K}_{l+1}$$

for some non-zero complex number η and some integer k, where $\mathbf{K}_m$ is the $m \times m$ skew-diagonal matrix

$$\mathbf{K}_m = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{35}$$

The second step is to note that with $\mathbf{S}(t)$ defined by

$$\mathbf{S}(t) = (1/\sqrt{2}) \begin{pmatrix} \mathbf{K}_{(l+1)/2} & \mathbf{1}_{(l+1)/2} \\ \mathbf{1}_{(l+1)/2} & -\mathbf{K}_{(l+1)/2} \end{pmatrix} \tag{36}$$

then

$$\mathbf{S}(t)\mathbf{K}_{l+1}\mathbf{S}(t)^{-1} = \begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{(l+1)/2} \end{pmatrix}$$

whose right-hand side is the $\mathbf{U}(t)$ matrix of (33), which is of the form (18).

2.4.2. $k_{(l+1)/2} + k_{(l+3)/2}$ assumed odd. Define $\mathbf{S}(t)$ to be the $(l+1) \times (l+1)$ diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} 1 & \text{for } j = (l+3)/2, \cdots, l+1 \\ (\eta_{l+2-j}/\eta_j)^{1/2} t^{(k_{l+2-j}-k_j-1)/2} & \text{for } j = 1, \cdots, (l+1)/2. \end{cases} \tag{37}$$

Then, with $\mathbf{U}(t)$ as defined in (30),

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t)^{-1} = \eta t^k \mathbf{U}'(t)$$

where

$$\mathbf{U}'(t) = \begin{pmatrix} \mathbf{0} & \mathbf{K}_{(l+1)/2} \\ t\mathbf{K}_{(l+1)/2} & \mathbf{0} \end{pmatrix} \tag{38}$$

and where η is some non-zero complex number and k is some integer. Thus *every* type 1a involutive automorphism associated with the matrix $\mathbf{U}(t)$ of (30) and $u = 1$ with $k_{(l+1)/2} + k_{(l+3)/2}$ assumed odd is conjugate to that associated with the matrix $\mathbf{U}'(t)$ of (38) and $u = 1$, which, by (I.67), (I.71), (I.73)) and (I.138), is

$$\begin{aligned}
\psi(h_{\alpha_0}) &= h_{\alpha_1} + h_{\alpha_2} + \cdots + h_{\alpha_l} \\
\psi(h_{\alpha_{(l+1)/2}}) &= h_{\alpha_0} + h_{\alpha_1} + \cdots + h_{\alpha_{(l-1)/2}} \\
&\qquad + h_{\alpha_{(l+3)/2}} + \cdots + h_{\alpha_l} \\
\psi(h_{\alpha_j}) &= -h_{\alpha_{l+1-j}} \qquad (j \neq 0, (l+1)/2) \\
\psi(c) &= c \\
\psi(d) &= (l+1) \sum_{j=1}^{(l-1)/2} j(h_{\alpha_j} + h_{\alpha_{l+1-j}}) + \{(l+1)^2/2\} h_{\alpha_{(l+1)/2}} \\
&\qquad - \{(l+1)^2/4\} c + d \\
\psi(e_{\pm\alpha_0}) &= -e_{\pm\{\alpha_1+\alpha_2+\cdots+\alpha_l\}} \\
\psi(e_{\pm\alpha_{(l+1)/2}}) &= -e_{\pm\{\alpha_0+\alpha_1+\cdots+\alpha_{(l-1)/2}+\alpha_{(l+3)/2}+\cdots+\alpha_l\}} \\
\psi(e_{\pm\alpha_j}) &= -e_{\mp\alpha_{l+1-j}} \qquad (j \neq 0, (l+1)/2).
\end{aligned} \tag{39}$$

2.5. Summary of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = 1$

The above considerations show that the number of conjugacy classes of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = 1$ is $\frac{1}{2}(l+2)$ if l is even and is $\frac{1}{2}(l+5)$ if l is odd. Their representatives may be taken to be

(i) the identity automorphism;

(ii) the $[\frac{1}{2}(l+1)]$ automorphisms (23), which correspond to the matrices $\mathbf{U}(t)$ of (18), where the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (22);

(iii) for l odd, the involutive automorphism (39), which corresponds to the matrix of (38).

3. Study of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$

3.1. Relevant conjugacy classes for the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$

The relevant conjugacy classes for the involutive automorphisms of type 1a with $u = -1$ are exactly the same as those for the involutive automorphisms of type 1a with $u = 1$ that were described in section 2.1. Consequently attention may again be concentrated on the involutive automorphisms associated with the Weyl transformations alone, and it is again convenient to divide the conjugacy classes of Weyl transformations into the three sets (A), (B) and (C) defined in section 2.1.

The type 1a involutive automorphisms with $u = -1$ corresponding to these sets will now be considered in turn.

3.2. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$ corresponding to set (A)*

The analysis for the $u = 1$ case may be repeated to show that the most general $(l+1) \times (l+1)$ matrix $\mathbf{U}(t)$ that satisfies (16) with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (17), where, for $j = 2, \ldots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.136) for $u = -1$ reduces to $\mathbf{U}(t)\mathbf{U}(-t) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which again implies that $k_j = 0$ and $\eta_j = \pm 1$ for $j = 2, \ldots, l+1$. That is, $\mathbf{U}(t)$ is again given by (18), where $\eta_j = \pm 1$ for $j = 2, \ldots, l+1$.

However, in contrast to the $u = 1$ case, when $u = -1$ *all* of the type 1a involutive automorphisms corresponding to these matrices of (18) are mutually conjugate. That is, there is *only one* conjugacy class of type 1a involutive automorphisms with $u = -1$ corresponding to the root transformation set (A).

The first stage in the proof of this assertion is to merely repeat the line of argument given in section 2.2 to show that all type 1a involutive automorphisms with $u = -1$ with the same value of $\{\mathrm{tr}\mathbf{U}(t)\}^{l+1}/\{\det \mathbf{U}(t)\}$ are conjugate. (This time there are $[\frac{1}{2}(l+3)]$ distinct values of $\{\mathrm{tr}\mathbf{U}(t)\}^{l+1}/\{\det \mathbf{U}(t)\}$ to be considered. These may be taken to correspond to

$$n_+ = [\tfrac{1}{2}l], [\tfrac{1}{2}l] + 1, \ldots, l \tag{40}$$

as for $u = -1$ the choice $\mathbf{U}(t) = \mathbf{1}$ does not correspond to the identity automorphism, but corresponds to an automorphism of order two.) The remaining step is to note that with $\mathbf{S}(t)$ defined by

$$\mathbf{S}(t) = \begin{pmatrix} \mathbf{0} & \mathbf{1}_{n_+ + 1} \\ t\mathbf{1}_{l-n_+} & \mathbf{0} \end{pmatrix} \tag{41}$$

then

$$\mathbf{S}(t)\mathbf{1}_{l+1}\mathbf{S}(-t)^{-1} = \begin{pmatrix} \mathbf{1}_{n_+ + 1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{l-n_+} \end{pmatrix}. \tag{42}$$

The conjugacy condition (I.158) then implies that type 1a involutive automorphisms with $u = -1$ corresponding to the matrix on the right-hand side of (42) is conjugate to the type 1a involutive automorphism with $u = -1$ corresponding to $\mathbf{1}_{l+1}$ for *every* value of n_+ given in (40).

A convenient choice for the set $\{\eta_1, \eta_2, \ldots, \eta_{l+1}\}$ is

$$\eta_j = \begin{cases} 1 & \text{for } j = 1, \ldots, l \\ -1 & \text{for } j = l+1. \end{cases} \tag{43}$$

By (I.67), (I.71), (I.73)) and (I.138) this corresponds to the involutive automorphism

$$\begin{aligned}
&\psi(h_{\alpha_j}) = h_{\alpha_j} \qquad (j = 0, \ldots, l) \\
&\psi(c) = c \\
&\psi(d) = d \\
&\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j = 0, 1, \ldots, l-1) \\
&\psi(e_{\pm\alpha_l}) = -e_{\pm\alpha_l}.
\end{aligned} \tag{44}$$

3.3. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$ corresponding to set (B)

It will now be shown that all the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$ corresponding to set (B) are actually conjugate to (44). To see this it will be sufficient to study an example that incorporates all the general features. Consider therefore the case $l = 4$ with conjugacy class representative $S^0_{\alpha_1^0}$. The most general 5×5 matrix $\mathbf{U}(t)$ that satisfies (24) with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (26), where, for $j = 1, \ldots, 4$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.136) for $u = -1$ again reduces to $\mathbf{U}(t)\mathbf{U}(-t) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies

$$k_2 = -k_1 \qquad k_3 = 0 \qquad k_4 = 0 \qquad \eta_2 = (-1)^{k_1}(\eta_1)^{-1}$$
$$\eta_3 = \pm 1 \qquad \eta_4 = \pm 1. \tag{45}$$

Let $\mathbf{S}(t)$ to be the 5×5 diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} (\eta_j)^{-1} t^{-k_j} & \text{for } j = 1 \\ 1 & \text{for } j = 2\text{–}5. \end{cases}$$

Then, for the matrix $\mathbf{U}(t)$ of (26) with parameters specified by (45),

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{46}$$

This is diagonalizable with eigenvalues ± 1, so that the diagonalized form is the same as that of (18), and as the matrix $\mathbf{S}(t)$ of the associated similarity transformation and its inverse both have entries that are all Laurent polynomials in t, it follows that every type 1a involutive automorphism (with $u = -1$) associated with the matrix $\mathbf{U}(t)$ of (26) with parameters specified by (45) is conjugate to the type 1a involutive automorphism (with $u = -1$) of (44) of the previous subsection.

On repeating this analysis for any conjugacy class representative of the set (B), one finds that each factor $S^0_{\alpha_j^0}$ that appears is associated with a 2×2 submatrix of $\mathbf{U}(t)$ of the form

$$\begin{pmatrix} 0 & \eta_j t^{k_j} \\ (\eta_j)^{-1}(-t)^{-k_j} & 0 \end{pmatrix}$$

which appears in the (j, j), $(j, j+1)$, $(j+1, j)$ and $(j+1, j+1)$ positions of $\mathbf{U}(t)$ (as in (26)). Similarly, each factor $S^0_{\alpha_j^0}$ that is missing from the product $S^0_{\alpha_1^0} \circ S^0_{\alpha_3^0} \circ \ldots$ is associated with a 2×2 diagonal submatrix of $\mathbf{U}(t)$ (in the (j, j), $(j, j+1)$, $(j+1, j)$ and $(j+1, j+1)$ positions) with diagonal entries ± 1. Moreover, with the assumptions made about the set (B), the $(l+1, l+1)$ element of $\mathbf{U}(t)$ *always* has the value 1. Consequently the matrix obtained from a transformation similar to that of (46) is diagonalizable with eigenvalues ± 1, so that the diagonalized form is the same as that of (18), and that the matrix $\mathbf{S}(t)$ of the associated similarity transformation and its inverse both have entries that are all Laurent polynomials in t. Consequently *every* type 1a involutive automorphisms (with $u = 1$) associated with a root transformation of the set (B) is conjugate to the type 1a involutive automorphism (with $u = -1$) of (44) of the previous subsection.

3.4. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$ corresponding to set (C)

With the assumption that l is *odd*, the most general $(l+1)\times(l+1)$ matrix $\mathbf{U}(t)$ that corresponds to $S^0_{\alpha_1^0} \circ S^0_{\alpha_3^0} \circ \ldots \circ S^0_{\alpha_l^0}$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by (28). It is again convenient to re-order the rows and columns of $\mathbf{U}$(t) so that $\mathbf{U}(t)$ is given by (29), which actually corresponds to the root transformation $\rho^0 \circ \tau^0_{\mathrm{Cartan}}$. It follows that the most general automorphism of type 1a with $u = -1$ associated with this root transformation may be taken to correspond to (30), where, for $j = 2, \ldots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.136) for $u = -1$ again reduces to $\mathbf{U}(t)\mathbf{U}(-t) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$\eta_{l+1} = \eta_2\eta_l(-1)^{k_2} = \eta_3\eta_{l-1}(-1)^{k_3} = \ldots = \eta_{(l+1)/2}\eta_{(l+3)/2}(-1)^{k_{(l+1)/2}} \tag{47}$$

and

$$k_{l+1} = k_2 + k_l = k_3 + k_{l-1} = \ldots = k_{(l+1)/2} + k_{(l+3)/2} \tag{48}$$

with k_{l+1} being required to be *even*. Then $\eta_2, \eta_3, \ldots, \eta_{(l+3)/2}$ can be taken to be arbitrary non-zero complex numbers and $k_2, k_3, \ldots, k_{(l+1)/2}$ can be taken to be arbitrary integers, but $k_{(l+3)/2}$ must be such that $k_{(l+1)/2} + k_{(l+3)/2}$ is *even*.

It will now be shown that *all* the type 1a involutive automorphisms associated with these $\mathbf{U}(t)$ matrices (and $u = -1$) are conjugate to that associated with $u = -1$ and a $\mathbf{U}(t)$ that is a special case of (18). To demonstrate this it will be sufficient to apply the conjugacy condition (I.158) to the simplest case where $l = 3$. The first step is to take $\mathbf{S}(t)$ to be the 4×4 diagonal matrix of (34). Then

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(-t)^{-1} = \eta t^k \begin{pmatrix} \mathbf{0} & \mathbf{K}_2 \\ (-1)^{(k_2-k_3)/2}\mathbf{K}_2 & \mathbf{0} \end{pmatrix} \tag{49}$$

for some non-zero complex number η and some integer k. As the involutive condition implies that $k_2 + k_3$ is even then $k_2 - k_3$ must also be even. If $(k_2 - k_3)/2$ is even the matrix on the right-hand side of (49) reduces to $\mathbf{K}_4$, and with $\mathbf{S}(t)$ defined by (36)

$$\mathbf{S}(t)\mathbf{K}_4\mathbf{S}(-t)^{-1} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_2 \end{pmatrix}$$

whose right-hand side is of the form (18). If $(k_2 - k_3)/2$ is odd, with $\mathbf{S}(t)$ defined by

$$\mathbf{S}(t) = (1/\sqrt{2}) \begin{pmatrix} t\mathbf{K}_{(l+1)/2} & \mathbf{1}_{(l+1)/2} \\ \mathbf{1}_{(l+1)/2} & -t\mathbf{K}_{(l+1)/2} \end{pmatrix} \tag{50}$$

then

$$\mathsf{S}(t)\mathsf{K}_{l+1}\mathsf{S}(-t)^{-1} = (t+t^{-1})\begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{(l+1)/2} \end{pmatrix}$$

where the matrix on the right-hand side is again of the form (18).

3.5. Summary of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$

The above considerations show there is only one conjugacy class of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1a with $u = -1$. Its representative may be the automorphism (44), which corresponds to the matrix $\mathsf{U}(t)$ of (18), where the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (43).

4. Study of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = 1$

4.1. Relevant conjugacy classes for the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = 1$

It is easily shown that there exists no non-singular $(l+1)\times(l+1)$ matrix $\mathsf{U}(t)$ that satisfies the conditions

$$\mathsf{U}(t)\{-\tilde{\mathsf{h}}^0_{\alpha_j^0}\}\mathsf{U}(t)^{-1} = \mathsf{h}^0_{\alpha_j^0} \tag{51}$$

for $j = 1, \ldots, l$. Consequently there are *no* type 1b involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ with $u = 1$ corresponding to the identity root transformation of (8). Similarly there are *no* type 1b involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ with $u = 1$ corresponding any root transformations that are members of Weyl group, so attention may be concentrated on those associated with products of the Weyl transformations and the Cartan involution τ^0_{Cartan}.

It is convenient to divide the conjugacy classes of root transformations into the following three sets:

(D) the conjugacy class consisting of τ^0_{Cartan} alone;

(E) the conjugacy classes with representatives

$$S^0_{\alpha_1^0} \circ \tau^0_{\mathrm{Cartan}},\ S^0_{\alpha_1^0} \circ S^0_{\alpha_3^0} \circ \tau^0_{\mathrm{Cartan}}, \ldots, S^0_{\alpha_1^0} \circ S^0_{\alpha_3^0} \circ \ldots \circ S^0_{\alpha_k^0} \circ \tau^0_{\mathrm{Cartan}}$$

with

$$k = \begin{cases} l-2 & \text{if } l \text{ is odd} \\ l-1 & \text{if } l \text{ is even} \end{cases} \tag{52}$$

(F) if l is *odd*, the conjugacy class with representative

$$S^0_{\alpha_1^0} \circ S^0_{\alpha_3^0} \circ \ldots \circ S^0_{\alpha_l^0} \circ \tau^0_{\mathrm{Cartan}}.$$

The type 1b involutive automorphisms with $u = 1$ corresponding to these sets will now be considered in turn.

4.2. *Involutive automorphisms of* $A_l^{(1)}$ *for* $l \geqslant 3$ *of type 1b with* $u = 1$ *corresponding to set (D)*

Because the matrices $\mathbf{h}^0_{\alpha^0_j}$ are diagonal (cf (5)), the set of conditions

$$\mathsf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha^0_j}\}\mathsf{U}(t)^{-1} = -\mathbf{h}^0_{\alpha^0_j} \tag{53}$$

for $j = 1, \ldots, l$, immediately reduces to the set (16). Consequently the analysis given in the first part of section 2.2 again applies, so the most general $(l+1) \times (l+1)$ matrix $\mathsf{U}(t)$ with both $\mathsf{U}(t)$ and $\mathsf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (17), where, for $j = 2, \ldots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.139) for $u = 1$ reduces to $\mathsf{U}(t)\tilde{\mathsf{U}}(t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which imposes *no* further constraints on the η_j and k_j (for $j = 2, \ldots, l+1$).

It will now be shown that

(i) if l is *even* then every involutive automorphism of type 1b with $u = 1$ corresponding to the root transformation (D) is conjugate to that generated by $\mathsf{U}(t) = \mathbf{1}_{l+1}$; and

(ii) if l is *odd* then every involutive automorphism of type 1b with $u = 1$ corresponding to the root transformation (D) is conjugate either to that generated by $\mathsf{U}(t) = \mathbf{1}_{l+1}$ or to that generated by

$$\mathsf{U}(t) = \begin{pmatrix} \mathbf{1}_l & \mathbf{0} \\ \mathbf{0} & t\mathbf{1}_1 \end{pmatrix}. \tag{54}$$

The proof of these results deals exclusively with matrices $\mathsf{U}(t)$ that are diagonal. Two sets of diagonal elements will be said to be *equivalent* if and only if the corresponding involutive automorphisms are conjugate. Clearly two sets of diagonal elements that merely differ in their ordering of elements are equivalent.

The first stage in the argument is to show that the set of diagonal elements of the diagonal matrix $\mathsf{U}(t)$ of (17) is equivalent to that with $\mathsf{U}(t)_{11} = 1$ and $\mathsf{U}(t)_{jj} = 1$ or t for each $j = 2, \ldots, l+1$. The second stage is to demonstrate that the sets $\{\ldots, 1, 1, t, \ldots\}$ and $\{\ldots, t, t, t, \ldots\}$ are equivalent. (Both of these results follow by an immediate generalization of arguments given for the $l = 2$ case in section 4.2 of paper III). The following two conclusions then follow by repeated application of these results:

(i) If the set of diagonal elements contains an *even* number entries 1 (with the other entries being t), then this set is equivalent to the set $\{t, t, \ldots, t\}$, which, by (I.166), is equivalent to the set $\{1, 1, \ldots, 1\}$.

(ii) If the set of diagonal elements contains an *odd* number entries 1 (with the other entries again being assumed to be t), then this set is equivalent to the set $\{1, t, t, \ldots, t\}$, which is equivalent to the set $\{t^2, t, t, \ldots, t\}$ (by a further application of the generalization of the argument given in section 4.2 of Paper III.) By (I.166) this is equivalent to the set $\{t, 1, 1, \ldots, 1\}$, which is equivalent to the set $\{1, 1, \ldots, 1, t\}$.

If l is *even* then the set of $l+1$ diagonal elements contains an odd number of members. If an even number of these have value 1 (with the other entries being t), then the previous conclusion (i) implies that the set is equivalent to $\{1, 1, \ldots, 1\}$. However if an odd number of these $l+1$ entries have value 1 (with the other entries being t), then there are an even number of t entries, which can be converted to an even number of 1 entries, so again conclusion (i) implies that the set is equivalent to $\{1,1,\ldots,1\}$.

If l is *odd* then the set of $l+1$ diagonal elements contains an even number of members. If an even number of these have value 1 (with the other entries being t), then (i) implies that the set is equivalent to $\{1,1,\ldots,1\}$. If an odd number of these $l+1$ entries have value 1 (with the other entries being t), then there are an odd number of t entries, which *cannot* converted by the above arguments to an even number of 1 entries, and then conclusion (ii) implies that the set is equivalent to $\{1,1,\ldots,1,t\}$.

These arguments imply immediately that the sets $\{1,1,\ldots,1\}$ and $\{1,1,\ldots,1,t\}$ are equivalent if l is *even*. However, if l is *odd* the sets $\{1,1,\ldots,1\}$ and $\{1,1,\ldots,1,t\}$ are *not* equivalent, for if they were equivalent, and $\mathbf{U}'(t)$ is the diagonal matrix with diagonal elements $\{1,1,\ldots,1,t\}$, then (I.166), (I.168), (I.170) and (I.172) would imply that there must exist a $(l+1)\times(l+1)$ matrix $\mathbf{S}(t)$ with entries that are Laurent polynomials in t such that $\mathbf{S}(t)\mathbf{1}_{l+1}\tilde{\mathbf{S}}(t) = \eta t^k\mathbf{U}'(t)$ for some non-zero complex number η and some integer k. This requires that $\{\det\mathbf{S}(t)\}^2 = (\eta t^k)^{l+1}t$, and as $\det\mathbf{S}(t)$ must be a Laurent polynomial in t, this is only possible if $\{\det\mathbf{S}(t)\}^2 = \eta' t^{k'}$ for some non-zero complex number η' and some integer k'. Then $k(l+1)+1 = 2k'$, so k' cannot be an integer if l is odd.

With $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = 1$ (I.68), (I.71) and (I.73) imply that the corresponding type 1b involutive automorphism is

$$\begin{aligned}
&\psi(h_{\alpha_0}) = h_{\alpha_0} + 2\sum_{j=1}^{l} h_{\alpha_j}\\
&\psi(h_{\alpha_j}) = -h_{\alpha_j} \qquad (j=1,\ldots,l)\\
&\psi(c) = c \qquad \psi(d) = d\\
&\psi(e_{\pm\alpha_0}) = e_{\pm\{\alpha_0+2\alpha_1+2\alpha_2+\ldots+2\alpha_l\}}\\
&\psi(e_{\pm\alpha_j}) = e_{\mp\alpha_j} \qquad (j=1,\ldots,l).
\end{aligned} \tag{55}$$

Similarly, with $\mathbf{U}(t)$ given by (54) and $u = 1$ (I.68), (I.71) and (I.73) imply that the corresponding type 1b involutive automorphism is

$$\begin{aligned}
&\psi(h_{\alpha_0}) = \sum_{j=1}^{l} h_{\alpha_j}\\
&\psi(h_{\alpha_j}) = -h_{\alpha_j} \qquad (j=1,\ldots,l-1)\\
&\psi(h_{\alpha_l}) = \sum_{j=0}^{l-1} h_{\alpha_j}\\
&\psi(c) = c \qquad \psi(d) = 2\sum_{j=1}^{l} j h_{\alpha_j} - lc + d
\end{aligned} \tag{56}$$

$$\psi(e_{\pm\alpha_0}) = e_{\pm\{\alpha_1+\alpha_2+\ldots+\alpha_l\}}$$

$$\psi(e_{\pm\alpha_j}) = e_{\mp\alpha_j} \qquad (j = 1,\ldots,l-1)$$

$$\psi(e_{\pm\alpha_l}) = e_{\pm\{\alpha_0+\alpha_1+\ldots+\alpha_{l-1}\}}.$$

4.3. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = 1$ corresponding to set (E)*

It will now be shown that all the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = 1$ corresponding to set (E) are actually conjugate to one of the set (D). To see this it will be sufficient to study an example that incorporates all the general features. Consider therefore the case $l = 4$ with conjugacy class representative $S^0_{\alpha_1^0}$. Because the matrices $\mathbf{h}^0_{\alpha_j^0}$ are diagonal (cf (5)), the set of conditions

$$\begin{aligned}
\mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_1^0}\}\mathbf{U}(t)^{-1} &= \mathbf{h}^0_{\alpha_1^0}\\
\mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_2^0}\}\mathbf{U}(t)^{-1} &= -\mathbf{h}^0_{\alpha_1^0} - \mathbf{h}^0_{\alpha_2^0}\\
\mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_3^0}\}\mathbf{U}(t)^{-1} &= -\mathbf{h}^0_{\alpha_3^0}\\
\mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha_4^0}\}\mathbf{U}(t)^{-1} &= -\mathbf{h}^0_{\alpha_4^0}
\end{aligned} \tag{57}$$

immediately reduces to the set (24). Consequently the analysis given in the first part of section 2.3 again applies, so the most general 5×5 matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (26), where, for $j = 1,\ldots,4$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.139) for $u = 1$ reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t)^{-1} = \eta t^k\mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies

$$\eta_1 = \eta_2 \qquad k_1 = k_2. \tag{58}$$

but, for $j = 2,3,4$, η_j and k_j do not have to satisfy any further constraints for. With $\mathbf{S}(t)$ defined by

$$\mathbf{S}(t) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{59}$$

it follows that with $\mathbf{U}(t)$ given by (26) (and subject to the constraints (58))

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t) = \begin{pmatrix} \eta_1 t^{k_1} & 0 & 0 & 0 & 0\\ 0 & -\eta_1 t^{k_1} & 0 & 0 & 0\\ 0 & 0 & \eta_3 t^{k_3} & 0 & 0\\ 0 & 0 & 0 & \eta_4 t^{k_4} & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{60}$$

Thus, by (I.166), the type 1b involutive automorphism with $u = 1$ and with $\mathbf{U}(t)$ given by (26) (and subject to the constraints (58)) is conjugate to that corresponding to the matrix on the right-hand side of (60), which is merely a rearrangement of a special case of (17).

On repeating this analysis for any conjugacy class representative of the set (E), one finds that after imposing the involutive condition that each factor $S^0_{\alpha^0_j}$ that appears is associated with a 2×2 submatrix of $\mathbf{U}(t)$ of the form

$$\begin{pmatrix} 0 & \eta_j t^{k_j} \\ \eta_j t^{k_j} & 0 \end{pmatrix}$$

which appears in the (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ positions of $\mathbf{U}(t)$. Similarly, each factor $S^0_{\alpha^0_j}$ that is missing from the product $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \ldots$ is associated with a 2×2 diagonal submatrix of $\mathbf{U}(t)$ (in the (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ positions) with diagonal entries $\eta_j t^{k_j}$ and $\eta_{j+1} t^{k_{j+1}}$. Moreover, with the assumptions made about the set (B), the $(l+1,l+1)$ element of $\mathbf{U}(t)$ *always* has the value 1. Then there always exists a matrix $\mathbf{S}(t)$ (with entries ±1) such that $\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t)$ is merely a rearrangement of a special case of (17), so no new classes of involutive automorphisms can appear.

4.4. Involutive automorphisms of $A^{(1)}_l$ for $l \geqslant 3$ of type 1b with $u = 1$ corresponding to set (F)

The assumption will be made in this subsection that l is *odd*, and it will be convenient to take ρ^0 as the class representative (in place of $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \ldots \circ S^0_{\alpha^0_l} \circ \tau^0_{\text{Cartan}}$). Because the matrices $\mathbf{h}^0_{\alpha^0_j}$ are diagonal (cf (5)), the set of conditions

$$\mathbf{U}(t)\{-\tilde{\mathbf{h}}^0_{\alpha^0_j}\}\mathbf{U}(t)^{-1} = \mathbf{h}^0_{\alpha^0_{l+j-1}}$$

for $j = 1, \ldots, l$, immediately reduces to the set

$$\mathbf{U}(t)\mathbf{h}^0_{\alpha^0_j}\mathbf{U}(t)^{-1} = -\mathbf{h}^0_{\alpha^0_{l+j-1}}$$

for $j = 1, \ldots, l$. Consequently the analysis given in the first part of section 2.4 again applies, so the most general $(l+1)\times(l+1)$ matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (30), where, for $j = 2, \ldots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.139) for $u = 1$ reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$\eta_{l+1} = \pm1 \qquad k_{l+1} = 0 \tag{61}$$

and

$$\eta_{l-j} = \eta_{j+2}\eta_{l+1} \qquad k_{l-j} = k_{j+2} \tag{62}$$

for $j = 0, 1, \ldots, \frac{1}{2}(l-3)$.

Define $\mathsf{S}(t)$ to be the $(l+1) \times (l+1)$ diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} 1 & \text{for } j = 1, l+1 \\ (\eta_j t^{k_j})^{-1/2} & \text{for } j = 2, \ldots, (l+1)/2. \\ (\eta_{l+2-j} t^{k_{l+2-j}})^{-1/2} & \text{for } j = (l+3)/2, \ldots, l. \end{cases}$$

Then

$$\mathsf{S}(t)\mathsf{U}(t)\tilde{\mathsf{S}}(t) = \begin{pmatrix} 0 & \mathsf{K}_{(l+1)/2} \\ \eta_{l+1}\mathsf{K}_{(l+1)/2} & 0 \end{pmatrix} \tag{63}$$

where $\mathsf{K}_{(l+1)/2}$ is defined in (35). There are two cases to be considered.

4.4.1. $\eta_{l+1} = 1$. In this case the matrix on the right-hand side of (63) reduces to the *symmetric* matrix K_{l+1}. With $\mathsf{S}(t)$ is defined by

$$\mathsf{S}(t) = \begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathsf{K}_{(l+1)/2} \\ \mathsf{K}_{(l+1)/2} & -\mathbf{1}_{(l+1)/2} \end{pmatrix} \tag{64}$$

then

$$\mathsf{S}(t)\mathsf{K}_{l+1}\tilde{\mathsf{S}}(t) = 2\begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{(l+1)/2} \end{pmatrix}.$$

As the matrix on the right-hand side here is merely a rearrangement of a special case of (17), no new classes of involutive automorphisms can appear in this case.

4.4.2. $\eta_{l+1} = -1$. In this case the matrix on the right-hand side of (63) is *antisymmetric*. Because of the symmetry and antisymmetry properties of the matrices involved, the conjugacy conditions (I.166), (I.168), (I.182) and (I.184) imply that the type1b involutive automorphism with $u = 1$ corresponding to this is *not* conjugate to that any of those of the set (D). With

$$\mathsf{U}(t) = \begin{pmatrix} 0 & \mathsf{K}_{(l+1)/2} \\ -\mathsf{K}_{(l+1)/2} & 0 \end{pmatrix} \tag{65}$$

and $u = 1$ (I.68), (I.71) and (I.73) imply that the corresponding type 1b involutive automorphism is

$$\begin{aligned} &\psi(h_{\alpha_0}) = h_{\alpha_0} \\ &\psi(h_{\alpha_j}) = h_{\alpha_{l+1-j}} \qquad (j = 1, \ldots, l) \\ &\psi(c) = c \qquad \psi(d) = d \\ &\psi(e_{\pm\alpha_0}) = -e_{\pm\alpha_0} \\ &\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_{l+1-j}} \qquad (j = 1, \ldots, \tfrac{1}{2}(l-1), \tfrac{1}{2}(l+3), \ldots, l) \\ &\psi(e_{\pm\alpha_j}) = -e_{\pm\alpha_{l+1-j}} \qquad (j = \tfrac{1}{2}(l+1)). \end{aligned} \tag{66}$$

4.5. Summary of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = 1$

The above considerations show that the number of conjugacy classes of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = 1$ is 1 if l is even and is 3 if l is odd. Their representatives may be taken to be

(i) for l even or odd, the involutive automorphism (55), which corresponds to the matrix $\mathbf{U}(t) = \mathbf{1}_{l+1}$;

(ii) for l odd, the involutive automorphism (56), which corresponds to the matrix $\mathbf{U}(t)$ of (54);

(iii) for l odd, the involutive automorphism (66), which corresponds to the matrix $\mathbf{U}(t)$ of (65).

5. Study of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$

5.1. Relevant conjugacy classes for the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$

The relevant conjugacy classes for the involutive automorphisms of type 1b with $u = -1$ are exactly the same as those for the involutive automorphisms of type 1b with $u = 1$ that were described in section 4.1. Consequently attention may again be concentrated on the involutive automorphisms associated with products of the Weyl transformations and the Cartan involution τ^0_{Cartan}, and it is again convenient to divide the conjugacy classes of root transformations into the three sets (D), (E) and (F) defined in section 4.1.

The type 1b involutive automorphisms with $u = -1$ corresponding to these sets will now be considered in turn.

5.2. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$ corresponding to set (D)

The set of conditions on the matrix is $\mathbf{U}(t)$ is again (53), which again immediately reduces to the set (16). Consequently the analysis given in the first part of section 2.2 applies yet again, so the most general $(l + 1) \times (l + 1)$ matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (17), where, for $j = 2, \ldots, l + 1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.248) for $u = -1$ now reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(-t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which imposes *no* further constraints on the η_j (for $j = 2, \ldots, l + 1$) but requires that every k_j must be *even* (for $j = 2, \ldots, l + 1$).

The arguments given in section 4.2 can be repeated to show that every involutive automorphism of type 1b with $u = -1$ corresponding to the root transformation (D) is conjugate to that generated by $\mathbf{U}(t) = \mathbf{1}_{l+1}$. (The other possibility considered in section 4.2 of having an involutive automorphism of type 1b with $\mathbf{U}(t)$ defined by (54) is excluded here by the requirement that the k_j must be even.)

With $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = -1$ (I.68), (I.71) and (I.73) imply that the corresponding type 1b involutive automorphism is

$$\psi(h_{\alpha_0}) = h_{\alpha_0} + 2 \sum_{j=1}^{l} h_{\alpha_j}$$

$$\psi(h_{\alpha_j}) = -h_{\alpha_j} \qquad (j = 1, \ldots, l)$$
$$\psi(c) = c \qquad \psi(d) = d$$
$$\psi(e_{\pm\alpha_0}) = -e_{\pm\{\alpha_0+2\alpha_1+2\alpha_2+\ldots+2\alpha_l\}}$$
$$\psi(e_{\pm\alpha_j}) = e_{\mp\alpha_j} \qquad (j = 1, \ldots, l). \tag{67}$$

5.3. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$ corresponding to set (E)*

It will now be shown that all the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$ corresponding to set (E) are actually conjugate to (67). To demonstrate this it will again be sufficient to study an example that incorporates all the general features. Consider therefore the case $l = 4$ with conjugacy class representative $S^0_{\alpha_1^0}$. The set of conditions (57) immediately reduces to the set (24). Consequently the analysis given in the first part of section 2.3 again applies, so the most general 5×5 matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (26), where, for $j = 1, \ldots, 4$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.139) for $u = -1$ reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(-t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies

$$\eta_1 = \eta_2(-1)^{k_2} \qquad k_1 = k_2 \tag{68}$$

with η_2, η_3, η_4 and k_2 experiencing no further restrictions, but k_3 and k_4 being required to be *even*. If k_2 is even, then with $\mathbf{S}(t)$ defined by (59) it follows that with $\mathbf{U}(t)$ given by (56) (and subject to the constraints (68))

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(-t) = \begin{pmatrix} \eta_2 t^{k_2} & 0 & 0 & 0 & 0 \\ 0 & -\eta_2 t^{k_2} & 0 & 0 & 0 \\ 0 & 0 & \eta_3 t^{k_3} & 0 & 0 \\ 0 & 0 & 0 & \eta_4 t^{k_4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{69}$$

where all the powers of t that appear are *even*. Likewise, if k_2 is odd, then with $\mathbf{S}(t)$ defined by

$$\mathbf{S}(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ t & -t^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

it follows that with $\mathbf{U}(t)$ given by (26) (and subject to the constraints (68))

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(-t) = \begin{pmatrix} 2\eta_2 t^{k_2+1} & 0 & 0 & 0 & 0 \\ 0 & 2\eta_2 t^{k_2+3} & 0 & 0 & 0 \\ 0 & 0 & \eta_3 t^{k_3} & 0 & 0 \\ 0 & 0 & 0 & \eta_4 t^{k_4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where again all the powers of t that appear are *even*. Thus, in both cases, by (I.166), the type 1b involutive automorphism with $u = -1$ and with $\mathbf{U}(t)$ given by (26) (and subject to the constraints (68)) is conjugate to that corresponding to the matrix that is merely a rearrangement of a special case of (17).

On repeating this analysis for any conjugacy class representative of the set (E), one finds (after imposing the involutive condition) that each factor $S^0_{\alpha^0_j}$ that appears is associated with a 2×2 submatrix of $\mathbf{U}(t)$ of the form

$$\begin{pmatrix} 0 & \eta_{j+1} t^{k_{j+1}} \\ \eta_{j+1}(-t)^{k_{j+1}} & 0 \end{pmatrix}$$

which appears in the (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ positions of $\mathbf{U}(t)$. Similarly, each factor $S^0_{\alpha^0_j}$ that is missing from the product $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \dots$ is associated with a 2×2 diagonal submatrix of $\mathbf{U}(t)$ (in the (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ positions) with diagonal entries $\eta_j t^{k_j}$ and $\eta_{j+1} t^{k_{j+1}}$ and with k_j and k_{j+1} even. Moreover, with the assumptions made about the set (B), the $(l+1,l+1)$ element of $\mathbf{U}(t)$ *always* has the value 1. Then there always exists a matrix $\mathbf{S}(t)$ (with entries ± 1) such that $\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(-t)$ is merely a rearrangement of a special case of (17), so no new classes of involutive automorphisms can appear.

5.4. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$ corresponding to set (F)

It will now be shown that all the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$ corresponding to set (F) are actually conjugate to (67). The assumption will be made in this subsection that l is *odd*, and it will be convenient to take ρ^0 as the class representative (in place of $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \dots \circ S^0_{\alpha^0_l} \circ \tau^0_{\mathrm{Cartan}}$). Again the analysis given in the first part of section 2.4 again applies, so the most general $(l+1) \times (l+1)$ matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (30), where, for $j = 2, \dots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.139) for $u = -1$ reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(-t)^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$\eta_{l+1} = \pm 1 \qquad k_{l+1} = 0 \tag{70}$$

and

$$\eta_{l-j} = \eta_{j+2}\eta_{l+1}(-1)^{k_{j+2}} \qquad k_{l-j} = k_{j+2} \tag{71}$$

for $j = 0, 1, \dots, \frac{1}{2}(l-3)$.

Define $\mathbf{S}(t)$ to be the $(l+1) \times (l+1)$ diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} 1 & \text{for } j = 1, 2, \dots, (l+1)/2 \text{ and } l+1 \\ \{\eta_{l+2-j}(-t)^{k_{l+2-j}}\}^{-1} & \text{for } j = (l+3)/2, \dots, l. \end{cases}$$

Then

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(-t) = \begin{pmatrix} 0 & \mathbf{K}_{(l+1)/2} \\ \eta_{l+1}\mathbf{K}_{(l+1)/2} & 0 \end{pmatrix}$$

where $\mathsf{K}_{(l+1)/2}$ is defined in (35). If $\eta_{l+1} = 1$ and $\mathsf{S}(t)$ is defined by (64), then

$$\mathsf{S}(t) \begin{pmatrix} \mathbf{0} & \mathsf{K}_{(l+1)/2} \\ \eta_{l+1}\mathsf{K}_{(l+1)/2} & \mathbf{0} \end{pmatrix} \tilde{\mathsf{S}}(-t) = 2 \begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{(l+1)/2} \end{pmatrix}.$$

Similarly, if $\eta_{l+1} = -1$ and $\mathsf{S}(t)$ is defined by

$$\mathsf{S}(t) = \begin{pmatrix} \mathbf{1}_{(l+1)/2} & t\mathsf{K}_{(l+1)/2} \\ t\mathsf{K}_{(l+1)/2} & -t^2\mathbf{1}_{(l+1)/2} \end{pmatrix}$$

then

$$\mathsf{S}(t) \begin{pmatrix} \mathbf{0} & \mathsf{K}_{(l+1)/2} \\ \eta_{l+1}\mathsf{K}_{(l+1)/2} & \mathbf{0} \end{pmatrix} \tilde{\mathsf{S}}(-t) = -2t \begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & t^2\mathbf{1}_{(l+1)/2} \end{pmatrix}.$$

The matrix on the right-hand side here yields the same involutive automorphism as the matrix

$$\begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & t^2\mathbf{1}_{(l+1)/2} \end{pmatrix}.$$

Thus, in *both* cases, by (I.166), the type 1b involutive automorphism with $u = -1$ and with $\mathsf{U}(t)$ given by (30) (and subject to the constraints (70) and (71)) is conjugate to that corresponding to a matrix that is merely a rearrangement of a special case of (17).

5.5. Summary of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$

The above considerations show that there is only one conjugacy class of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 1b with $u = -1$. Its representative may be taken to be the involutive automorphism (67), which corresponds to the matrix $\mathsf{U}(t) = \mathbf{1}_{l+1}$.

6. Study of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a

6.1. Relevant conjugacy classes for the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a

The relevant conjugacy classes for the involutive automorphisms of type 2a are exactly the same as those for the involutive automorphisms of type 1a with $u = 1$ that were described in section 2.1. Consequently attention may again be concentrated on the involutive automorphisms associated with the Weyl transformations alone, and it is again convenient to divide the conjugacy classes of Weyl transformations into the three sets (A), (B) and (C) defined in section 2.1.

The type 2a involutive automorphisms with $u = 1$ corresponding to these sets will now be considered in turn.

6.2. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a with $u = 1$ corresponding to set (A)

The analysis for the type 1a case with $u = 1$ may be repeated to show that the most general $(l+1) \times (l+1)$ matrix $\mathbf{U}(t)$ that satisfies (16) with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (17), where, for $j = 2, \ldots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.142) for $u = 1$ reduces to $\mathbf{U}(t)\mathbf{U}(t^{-1}) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that $\eta_j = \pm 1$ (for $j = 2, \ldots, l+1$), with no further restrictions being imposed on the k_j (for $j = 2, \ldots, l+1$). With $\mathbf{S}(t)$ defined to be the $(l+1) \times (l+1)$ diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} 1 & \text{for } j = 1 \\ t^{-\kappa_j} & \text{for } j = 2, \ldots, l+1 \end{cases}$$

then

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \eta_2 t^{k_2 - 2\kappa_2} & 0 & \cdots & 0 \\ 0 & 0 & \eta_3 t^{k_3 - 2\kappa_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \eta_{l+1} t^{k_{l+1} - 2\kappa_{l+1}} \end{pmatrix}$$

so the conjugacy condition (I.174) implies that every type 2a involutive automorphism (with $u = 1$) is conjugate to one with $\mathbf{U}(t)$ given by (17), but where it may be assumed that

$$\eta_j = \pm 1 \qquad k_j = 0 \text{ or } 1 \tag{72}$$

for $j = 2, \ldots, l+1$. There are two main cases to be considered.

6.2.1. $k_j = 0$ for every $j = 2, 3, \ldots, l$. In this case the analysis is essentially the same as that for the corresponding type 1a automorphisms with $u = 1$ that was given in section 2.2, the only difference being that $\mathbf{U}(t) = \mathbf{1}_{l+1}$ now corresponds not to the identity automorphism but to an involutive automorphism of order 2. With $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = 1$ the corresponding type 2a involutive automorphism is (by (I.69), (I.71), (I.73)) and (I.143)):

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} - 2\sum_{j=1}^{l} h_{\alpha_j} \\ &\psi(h_{\alpha_j}) = h_{\alpha_j} \qquad (j = 1, \ldots, l) \\ &\psi(c) = -c \\ &\psi(d) = d \\ &\psi(e_{\pm\alpha_0}) = e_{\mp\{\alpha_0 + 2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_l\}} \\ &\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j = 1, \ldots, l). \end{aligned} \tag{73}$$

With $\mathsf{U}(t)$ corresponding to

$$\eta_j = \begin{cases} 1 & \text{for } j = 1, \ldots, n_+ + 1 \\ -1 & \text{for } j = n_+ + 2, \ldots, l+1 \end{cases} \tag{74}$$

and $u = 1$, where

$$n_+ = [\tfrac{1}{2}l] \qquad [\tfrac{1}{2}l] + 1, \ldots, l-1 \tag{75}$$

(with $[a]$ again denoting the largest integer not exceeding a), the corresponding type 2a involutive automorphism is (by (I.69), (I.71), (I.73)) and (I.143))

$$\begin{aligned} &\psi(h_{\alpha_0}) = -h_{\alpha_0} - 2\sum_{j=1}^{l} h_{\alpha_j} \\ &\psi(h_{\alpha_j}) = h_{\alpha_j} \qquad (j = 1, \ldots, l) \\ &\psi(c) = -c \\ &\psi(d) = d \\ &\psi(e_{\pm\alpha_0}) = -e_{\mp\{\alpha_0 + 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l\}} \\ &\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j = 1, \ldots, n_+) \\ &\psi(e_{\pm\alpha_j}) = -e_{\pm\alpha_j} \qquad (j = n_+ + 1) \\ &\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j = n_+ + 2, \ldots, l). \end{aligned} \tag{76}$$

6.2.2. At least one of the k_j (for $j = 2, 3, \ldots, l$) takes the value 1. Let N_+, N_-, and N_t be the number of diagonal entries in the $\mathsf{U}(t)$ matrix of (17) with values $+1$, -1 and $\pm t$ respectively. With the assumption that has just been made $N_t \geqslant 1$. There are two subcases to examine:

(a) $N_+ = N_-$ (and $N_t \geqslant 1$). In this subcase every involutive automorphism is conjugate to (73) or one of (76). The proof of this assertion is as follows. Clearly the ± 1 and $\pm t$ entries can be reordered by an appropriate similarity transformation (with $\mathsf{S}(t)$ independent of t) to give

$$\mathsf{U}(t) = \operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{N_+}, \underbrace{-1, -1, \ldots, -1}_{N_-}, \underbrace{\pm t, \pm t, \ldots, \pm t}_{N_t}\}.$$

However the conjugacy condition (I.174) (with $\mathsf{S}(t) = 1_{l+1}$, $s = -1$, and $s^{-2}u_2 = 1$) implies that $\mathsf{U}(t)$ and $\mathsf{U}(-t)$ give conjugate automorphisms. Consequently at least one of the $\pm t$ entries may be taken to be t. Assuming that the ordering is such that this is the first,

$$\mathsf{U}(t) = \operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{N_+}, \underbrace{-1, -1, \ldots, -1}_{N_-}, \underbrace{t, \pm t, \ldots, \pm t}_{N_t}\}. \tag{77}$$

The argument given in the corresponding $l = 2$ case (in section 6.2 of paper III) can easily be extended to show that every $(1, -1, t)$ triple is equivalent to a $(t, -t, t)$

triple, so, by N_- applications, the automorphism corresponding to (77) is conjugate to that corresponding to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{t, t, \dots, t}_{N_+ = N_-}, \underbrace{-t, -t, \dots, -t}_{N_-}, \underbrace{t, \pm t, \dots, \pm t}_{N_t}\}.$$

This is conjugate to that with

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, 1, \dots, 1}_{N_+ = N_-}, \underbrace{-1, -1, \dots, -1}_{N_-}, \underbrace{1, \pm 1, \dots, \pm 1}_{N_t}\}$$

which is one of the set considered in the previous case. Thus no new involutive automorphisms appear when $N_+ = N_-$.

(b) $N_+ \neq N_-$ (and $N_t \geqslant 1$). In this subcase every involutive automorphism is conjugate to that associated with

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, 1, \dots, 1}_{N_+}, \underbrace{-1, -1, \dots, -1}_{N_-}, \underbrace{t, t, \dots, t}_{N_t}\} \tag{78}$$

where

$$[\tfrac{1}{2}l] + 1 \leqslant N_+ \leqslant l \tag{79}$$

$$0 \leqslant N_- \leqslant l - [\tfrac{1}{2}l] - 1 \tag{80}$$

$$1 \leqslant N_t \leqslant l - [\tfrac{1}{2}l] \tag{81}$$

$$N_+ + N_- + N_t = l + 1 \tag{82}$$

and *all* the terms involving the power t have the *positive* coefficient 1.

The proof of this assertion is as follows. There are two possibilities to be considered:

(1) $N_- > 0$. Suppose first that $N_- > N_+$. The argument as far as (77) is the same as before. Applying the result that every $(1, -1, t)$ triple is equivalent to a $(t, -t, t)$ triple N_+ times, one finds that the automorphism corresponding to (77) is conjugate to that corresponding to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{t, \dots, t}_{N_+}, \underbrace{-t, \dots, -t}_{N_+}, \underbrace{-1, \dots, -1}_{N_- - N_+}, \underbrace{t, \pm t, \dots, \pm t}_{N_t}\}$$

which is equivalent to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{t, \dots, t}_{N_+}, \underbrace{-t, \dots, -t}_{N_+}, \underbrace{-t^2, \dots, -t^2}_{N_- - N_+}, \underbrace{t, \pm t, \dots, \pm t}_{N_t}\}$$

which in turn is equivalent to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, \dots, 1}_{N_+}, \underbrace{-1, \dots, -1}_{N_+}, \underbrace{-t, \dots, -t}_{N_- - N_+}, \underbrace{1, \pm 1, \dots, \pm 1}_{N_t}\}.$$

As $\mathbf{U}(t)$ and $\mathbf{U}(-t)$ give conjugate automorphisms, this is equivalent to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1,\ldots,1}_{N_+},\underbrace{-1,\ldots,-1}_{N_+},\underbrace{t,\ldots,t}_{N_- - N_+},\underbrace{1,\pm 1,\ldots,\pm 1}_{N_t}\} \tag{83}$$

where *all* the terms involving the power t have the *positive* coefficient 1. If $0 < N_- < N_+$ a similar argument shows that the original $\mathbf{U}(t)$ is equivalent to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1,\ldots,1}_{N_-},\underbrace{t,\ldots,t}_{N_+ - N_-},\underbrace{-1,\ldots,-1}_{N_-},\underbrace{1,\pm 1,\ldots,\pm 1}_{N_t}\} \tag{84}$$

where again *all* the terms involving the power t have the *positive* coefficient 1. Both (83) and (84) have the form

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1,\ldots,1}_{N'_+},\underbrace{-1,\ldots,-1}_{N'_-},\underbrace{t,\ldots,t}_{N'_t}\} \tag{85}$$

for some values of N'_+, N'_-, and N'_t. If $N'_t \leqslant l - [\frac{1}{2}l]$ the only matter to be considered is the values of N'_+ and N'_-. By repeating the arguments used in the case when $k_j = 0$ for every $j = 2, 3, \ldots, l$, it follows that every $\mathbf{U}(t)$ is equivalent to one with $[\frac{1}{2}l] + 1 \leqslant N'_+$, so the stated result is obtained. On the other hand if $N'_t > l - [\frac{1}{2}l]$ a repeat of the argument leading to (83) shows that every $\mathbf{U}(t)$ is equivalent to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1,\ldots,1}_{N''_+},\underbrace{-1,\ldots,-1}_{N''_-},\underbrace{t,\ldots,t}_{N''_t}\}$$

with $N''_t = N'_+ - N'_-$, so $N''_t \leqslant l - [\frac{1}{2}l]$, and one is back with the previous situation.
(2) $N_- = 0$. In this case one can start with (85), which reduces to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1,1,\ldots,1}_{N_+},\underbrace{t,\pm t,\ldots,\pm t}_{N_t}\}. \tag{86}$$

Application of the previous arguments show that this is equivalent first to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{t,t,\ldots,t}_{N_+},\underbrace{t^2,\pm t^2,\ldots,\pm t^2}_{N_t}\}$$

and then to

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{t,t,\ldots,t}_{N_+},\underbrace{1,\pm 1,\ldots,\pm 1}_{N_t}\}. \tag{87}$$

Clearly, if at least one term in (86) is $-t$, then at least one -1 term appears in (87), and one is back to the previous case of $N_- \geqslant 1$. On the other hand , if all the powers of t in (86) have coefficient 1, either $N_+ \geqslant N_-$, when the desired result is obtained immediately, or $N_+ < N_-$, when (87) has the same form, but with N_+ and N_t interchanged, so one is back to the previous case.

It is quite easily shown using the conditions (I.174), (I.176), (I.178) and (I.180) that no two involutive automorphisms corresponding to $u = 1$ and to different matrices $\mathbf{U}(t)$ that are given by (78) with the constraints (79), (80), (81) and (82) all satisfied (and with $N_+ \neq N_-$) are conjugate.

The type 2a involutive automorphisms corresponding to $u = 1$ and the matrices $\mathbf{U}(t)$ of (78) can be divided into two sets.

(i) Those for which $N_- = 0$. For these

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, 1, \ldots, 1}_{N_+}, \underbrace{t, t, \ldots, t}_{N_t}\} \tag{88}$$

with

$$[\tfrac{1}{2}l] + 1 \leqslant N_+ \tag{89}$$

and

$$N_t = l + 1 - N_+. \tag{90}$$

Thus

$$\eta_j = 1 \tag{91}$$

for $j = 1, \ldots, l+1$, and

$$k_j = \begin{cases} 0 & \text{for } j = 1, \ldots, N_+ \\ 1 & \text{for } j = N_+ + 1, \ldots, l+1. \end{cases} \tag{92}$$

The corresponding type 2a involutive automorphism is (by (I.69), (I.71), (I.73)) and (I.143))

$$\begin{aligned}
&\psi(h_{\alpha_j}) = -\sum_{j=1}^{l} h_{\alpha_j} \qquad (j = 0) \\
&\psi(h_{\alpha_j}) = h_{\alpha_j} \qquad (j \neq 0, N_+) \\
&\psi(h_{\alpha_j}) = -\sum_{j=0}^{N_+-1} h_{\alpha_j} - \sum_{j=N_++1}^{l} h_{\alpha_j} \qquad (j = N_+) \\
&\psi(c) = -c \\
&\psi(d) = -2(l+1-N_+)\sum_{j=1}^{N_+} j h_{\alpha_j} - 2N_+ \sum_{j=1}^{l-N_+} j h_{\alpha_{l+1-j}} - 2(l+1-N_+)N_+ c - d \\
&\psi(e_{\pm\alpha_j}) = e_{\mp\{\alpha_1+\alpha_2+\ldots+\alpha_l\}} \qquad (j = 0) \\
&\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j \neq 0, N_+) \\
&\psi(e_{\pm\alpha_j}) = e_{\mp\{\alpha_0+\alpha_1+\ldots+\alpha_{N_+-1}+\alpha_{N_++1}+\ldots+\alpha_l\}} \qquad (j = N_+).
\end{aligned} \tag{93}$$

The number of conjugacy classes of this kind is $\frac{1}{2}l$ if l is even and $\frac{1}{2}(l+1)$ if l is odd.

(ii) Those for which $N_- \geqslant 1$. For these $\mathbf{U}(t)$ is given by (78)) with N_+, N_-, and N_t satisfying (79), (80) and (81). Thus

$$\eta_j = \begin{cases} 1 & \text{for } j = 1, \ldots, N_+ \\ -1 & \text{for } j = N_+ + 1, \ldots, N_+ + N_- \\ 1 & \text{for } j = N_+ + N_- + 1, \ldots, l+1 \end{cases} \tag{94}$$

and

$$k_j = \begin{cases} 0 & \text{for } j = 1, \ldots, N_+ + N_- \\ 1 & \text{for } j = N_+ + N_- + 1, \ldots, l+1. \end{cases} \tag{95}$$

The corresponding type 2a involutive automorphism is (by (I.69), (I.71), (I.73)) and (I.143))

$$\psi(h_{\alpha_j}) = -\sum_{j=1}^{l} h_{\alpha_j} \qquad (j=0)$$

$$\psi(h_{\alpha_j}) = h_{\alpha_j}, (j \neq 0, N_+ + N_-)$$

$$\psi(h_{\alpha_j}) = -\sum_{j=0}^{N_+ + N_- - 1} h_{\alpha_j} - \sum_{j=N_+ + N_- + 1}^{l} h_{\alpha_j} \qquad (j = N_+ + N_-)$$

$$\psi(c) = -c$$

$$\psi(d) = -2(l+1-N_+-N_-) \sum_{j=1}^{N_+ + N_-} j h_{\alpha_j} - 2N_+ \times \sum_{j=1}^{l-N_+-N_-} j h_{\alpha_{l+1-j}} - 2(l+1-N_+-N_-)(N_+ + N_-)c - d \tag{96}$$

$$\psi(e_{\pm\alpha_j}) = e_{\mp\{\alpha_1+\alpha_2+\cdots+\alpha_l\}} \qquad (j=0)$$

$$\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j = 1, \ldots, N_+ - 1)$$

$$\psi(e_{\pm\alpha_j}) = -e_{\pm\alpha_j} \qquad (j = N_+)$$

$$\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j = N_+ + 1, \ldots, N_+ + N_- + 1)$$

$$\psi(e_{\pm\alpha_j}) = -e_{\mp\{\alpha_0+\alpha_1+\cdots+\alpha_{N_++N_--1}+\alpha_{N_++N_-+1}+\cdots+\alpha_l\}} (j = N_+ + N_-)$$

$$\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_j} \qquad (j = N_+ + N_- + 1, \ldots, l).$$

The number of conjugacy classes of this kind is $\frac{1}{8}l(l-2)$ if l is even and $\frac{1}{8}(l-1)(l+1)$ if l is odd.

6.3. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a with $u = 1$ corresponding to set (B)

It will now be shown that all the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a with $u = 1$ corresponding to set (B) are actually conjugate to those corresponding to the set (A) that were considered in the previous subsection. To see this it will be sufficient to study an example that incorporates all the general features. Consider therefore the case $l = 4$ with conjugacy class representative $S^0_{\alpha^0_1}$. The most general 5×5 matrix $\mathbf{U}(t)$ that satisfies (24) with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (26), where, for $j = 1, \ldots, 4$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.142) for $u = 1$ again reduces to $\mathbf{U}(t)\mathbf{U}(t^{-1}) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies

$$\begin{aligned} &k_2 = k_1 \qquad k_3 = 0 \qquad k_4 = 0 \\ &\eta_2 = (\eta_1)^{-1} \qquad \eta_3 = \pm 1 \qquad \eta_4 = \pm 1. \end{aligned} \tag{97}$$

Let $\mathbf{S}(t)$ be the 5×5 matrix

$$\mathbf{S}(t) = \begin{pmatrix} 1 & \eta_1 & 0 & 0 & 0 \\ -(\eta_1)^{-1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{98}$$

Then, for the matrix $\mathbf{U}(t)$ of (26) with parameters specified by (97)

$$\mathbf{S}(t)\mathbf{U}(t)\mathbf{S}(t^{-1})^{-1} = \begin{pmatrix} t^{k_1} & 0 & 0 & 0 & 0 \\ 0 & -t^{k_1} & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{99}$$

which is a rearrangement of a special form of the general matrix $\mathbf{U}(t)$ considered in the previous subsection. As the matrix $\mathbf{S}(t)$ of the associated similarity transformation and its inverse both have entries that are all Laurent polynomials in t, it follows from (I.174) that every type 2a involutive automorphism (with $u = 1$) associated with the matrix $\mathbf{U}(t)$ of (26) with parameters specified by (97) is conjugate to a type 2a involutive automorphism (with $u = 1$) of the previous subsection.

On repeating this analysis for any conjugacy class representative of the set (B), one finds that each factor $S^0_{\alpha^0_j}$ that appears is associated with a 2×2 submatrix of $\mathbf{U}(t)$ of the form

$$\begin{pmatrix} 0 & \eta_j t^{k_j} \\ (\eta_j)^{-1} t^{k_j} & 0 \end{pmatrix}$$

which appears in the (j, j), $(j, j+1)$, $(j+1, j)$ and $(j+1, j+1)$ positions of $\mathbf{U}(t)$ (as in (26)). Similarly, each factor $S^0_{\alpha^0_j}$ that is missing from the product $S^0_{\alpha^0_1} \circ S^0_{\alpha^0_3} \circ \ldots$

is associated with a 2×2 diagonal submatrix of $\mathsf{U}(t)$ (in the (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ positions) with diagonal entries ± 1. Moreover, with the assumptions made about the set (B), the $(l+1,l+1)$ element of $\mathsf{U}(t)$ *always* has the value 1. Consequently the matrix obtained from a transformation similar to that of (99) is a rearrangement of a special form of the general matrix $\mathsf{U}(t)$ considered in the previous subsection. Consequently *every* type 2a involutive automorphisms (with $u = 1$) associated with a root transformation of the set (B) is conjugate to a type 2a involutive automorphism (with $u = 1$) of the previous subsection.

6.4. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a with $u = 1$ corresponding to set (C)*

With the assumption that l is *odd*, the most general $(l+1) \times (l+1)$ matrix $\mathsf{U}(t)$ that corresponds to $S^0_{\alpha_1^0} \circ S^0_{\alpha_3^0} \circ \ldots \circ S^0_{\alpha_l^0}$ with both $\mathsf{U}(t)$ and $\mathsf{U}(t)^{-1}$ having entries that are Laurent polynomials in t is given by (28). It is again convenient to re-order the rows and columns of U(t) so that $\mathsf{U}(t)$ is given by (29), which actually corresponds to the root transformation $\rho^0 \circ \tau^0_{\mathrm{Cartan}}$. It follows that the most general automorphism of type 2a with $u = 1$ associated with this root transformation may be taken to correspond to (30), where, for $j = 2, \ldots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.142) for $u = 1$ again reduces to $\mathsf{U}(t)\mathsf{U}(t^{-1}) = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$\eta_{l+1} = \eta_2\eta_l = \eta_3\eta_{l-1} = \ldots = \eta_{(l+1)/2}\eta_{(l+3)/2} \tag{100}$$

and

$$k_{l+1} = 0 \qquad k_l = k_2 \qquad k_{l-1} = k_3, \ldots, k_{(l+3)/2} = k_{(l+1)/2}. \tag{101}$$

Then $\eta_2, \eta_3, \ldots, \eta_{(l+3)/2}$ can be taken to be arbitrary non-zero complex numbers and $k_2, k_3, \ldots, k_{(l+1)/2}$ can be taken to be arbitrary integers.

It will now be shown that *all* the type 2a involutive automorphisms associated with these $\mathsf{U}(t)$ matrices (and $u = 1$) are conjugate to that associated with $u = 1$ and a $\mathsf{U}(t)$ that is a special case of those considered in subsection 6.2. To demonstrate this it will be sufficient to apply the conjugacy condition (I.174) to the simplest case where $l = 3$. The first step is to take $\mathsf{S}(t)$ to be the 4×4 diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} (\eta_2\eta_3)^{1/2} & \text{for } j = 1 \\ (\eta_3/\eta_2)^{1/2} t^{-k_2} & \text{for } j = 2 \\ 1 & \text{for } j = 3, 4. \end{cases}$$

Then

$$\mathsf{S}(t)\mathsf{U}(t)\mathsf{S}(t^{-1})^{-1} = (\eta_2\eta_3)^{1/2}\mathsf{K}_4. \tag{102}$$

However, with $\mathsf{S}(t)$ defined by (36)

$$\mathsf{S}(t)\mathsf{K}_4\mathsf{S}(t^{-1})^{-1} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_2 \end{pmatrix}$$

whose right-hand side is of the form (18).

6.5. Summary of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a

The above considerations show that the number of conjugacy classes of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2a is $\frac{1}{8}(l+2)(l+4)$ if l is even and is $\frac{1}{8}(l+3)(l+5)$ if l is odd. Their representatives may be taken to be

(i) the automorphism (73) which corresponds to $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = 1$;

(ii) the $l - [\frac{1}{2}l]$ automorphisms (76), which correspond to $u = 1$ and the matrices $\mathbf{U}(t)$ of (17), where $k_j = 0$ for every $j = 2, 3, \ldots, l$ and the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (74);

(iii) the $[\frac{1}{2}(l+1)]$ automorphisms (93), which correspond to $u = 1$ and the matrices $\mathbf{U}(t)$ of (88), where N_+ and N_t satisfy (89) and (90);

(iii) the automorphisms (93), which correspond to $u = 1$ and the matrices $\mathbf{U}(t)$ of (78)) with N_+, N_- and N_t satisfying (79), (80), (81) and (82) (the number of conjugacy classes of this kind being $\frac{1}{8}l(l-2)$ if l is even and $\frac{1}{8}(l-1)(l+1)$ if l is odd).

7. Study of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b

7.1. Relevant conjugacy classes for the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b with $u = 1$

The relevant conjugacy classes for the involutive automorphisms of type 2b with $u = 1$ are exactly the same as those for the involutive automorphisms of type 2b with $u = 1$ that were described in section 4.1. Consequently attention may again be concentrated on the involutive automorphisms associated with products of the Weyl transformations and the Cartan involution τ^0_{Cartan}, and it is again convenient to divide the conjugacy classes of root transformations into the three sets (D), (E) and (F) defined in section 4.1.

The type 2b involutive automorphisms with $u = 1$ corresponding to these sets will now be considered in turn.

7.2. Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b with $u = 1$ corresponding to set (D)

The set of conditions on the matrix is $\mathbf{U}(t)$ is again (53), which again immediately reduces to the set (16). Consequently the analysis given in the first part of section 2.2 applies yet again, so the most general $(l+1) \times (l+1)$ matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (17), where, for $j = 2, \ldots, l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.144) for $u = 1$ now reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t^{-1})^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which imposes *no* further constraints on the η_j (for $j = 2, \ldots, l+1$) but requires that $k_j = 0$ for $j = 2, \ldots, l+1$.

Define $\mathbf{S}(t)$ to be the $(l+1) \times (l+1)$ diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} 1 & \text{for } j = 1 \\ (\eta_j)^{-1/2} & \text{for } j = 2, \ldots, l+1. \end{cases}$$

Then

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t^{-1}) = \mathbf{1}_{l+1}$$

so the conjugacy condition (I.144) implies that every involutive automorphism of type 2b corresponding to the root transformation (D) is conjugate to that generated by $\mathbf{U}(t) = \mathbf{1}_{l+1}$ (and $u = 1$).

This involutive automorphism of type 2b corresponding to $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = 1$ is (by (I.70), (I.71), (I.73)) and (I.145)) the Cartan involution

$$\begin{aligned}
&\psi(h_{\alpha_j}) = -h_{\alpha_j} \qquad (j = 0, \ldots, l)\\
&\psi(c) = -c\\
&\psi(d) = -d\\
&\psi(e_{\pm\alpha_j}) = e_{\mp\alpha_j} \qquad (j = 0, \ldots, l).
\end{aligned} \tag{103}$$

7.3. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b with $u = 1$ corresponding to set (E)*

It will now be shown that all the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b with $u = 1$ corresponding to the set (E) are actually conjugate to those corresponding to the set (D). To demonstrate this it will again be sufficient to study an example that incorporates all the general features. Consider therefore the case $l = 4$ with conjugacy class representative $S^0_{\alpha_1^0}$. The set of conditions (57) immediately reduces to the set (24). Consequently the analysis given in the first part of section 2.3 again applies, so the most general 5×5 matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (26), where, for $j = 1, \ldots, 4$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.144) for $u = 1$ now reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t^{-1})^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies

$$\eta_1 = \eta_2 \qquad k_1 = -k_2 \qquad k_3 = 0 \qquad k_3 = 0 \tag{104}$$

with η_2, η_3, η_4, and k_2 experiencing no further restrictions.

With $\mathbf{S}(t)$ defined by

$$\mathbf{S}(t) = \begin{pmatrix} 1 & (2\eta_2)^{-1/2}t^{-k_2} & 0 & 0 & 0\\ (2\eta_2)^{-1/2}t^{k_2} & -1 & 0 & 0 & 0\\ 0 & 0 & (\eta_3)^{-1/2} & 0 & 0\\ 0 & 0 & 0 & (\eta_4)^{-1/2} & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

it follows that with $\mathbf{U}(t)$ given by (26) (and subject to the constraints (104))

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t^{-1}) = \mathrm{diag}\{1, -1, 1, 1, 1\}.$$

Thus, by (I.166), the type 2b involutive automorphism with $u = 1$ and with $\mathbf{U}(t)$ given by (26) (and subject to the constraints (104)) is conjugate to that corresponding to a matrix that is merely a special case that considered in the previous subsection.

On repeating this analysis for any conjugacy class representative of the set (E), one finds (after imposing the involutive condition) that each factor $S^0_{\alpha^0_j}$ that appears is associated with a 2×2 submatrix of $\mathbf{U}(t)$ of the form

$$\begin{pmatrix} 0 & \eta_{j+1}t^{-k_{j+1}} \\ \eta_j t^{k_{j+1}} & 0 \end{pmatrix}$$

which appears in the (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ positions of $\mathbf{U}(t)$. Similarly, each factor $S^0_{\alpha^0_j}$ that is missing from the product $S^0_{\alpha^0_1}\circ S^0_{\alpha^0_3}\circ\ldots$ is associated with a 2×2 diagonal submatrix of $\mathbf{U}(t)$ (in the (j,j), $(j,j+1)$, $(j+1,j)$ and $(j+1,j+1)$ positions) with diagonal entries η_j and η_{j+1}. Moreover, with the assumptions made about the set (B), the $(l+1,l+1)$ element of $\mathbf{U}(t)$ *always* has the value 1. Then there always exists a matrix $\mathbf{S}(t)$ whose entries are Laurent polynomials in t such that $\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(-t)$ is merely a special case of that considered in the previous subsection, so no new classes of involutive automorphisms can appear.

7.4. *Involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b with $u = 1$ corresponding to set (F)*

The assumption will be made in this subsection that l is *odd*, and it will be convenient to take ρ^0 as the class representative (in place of $S^0_{\alpha^0_1}\circ S^0_{\alpha^0_3}\circ\ldots\circ S^0_{\alpha^0_l}\circ\tau^0_{\text{Cartan}}$). Again the analysis given in the first part of section 2.4 again applies, so the most general $(l+1)\times(l+1)$ matrix $\mathbf{U}(t)$ with both $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ having entries that are Laurent polynomials in t may be taken to be given by (30), where, for $j = 2,\ldots,l+1$, η_j are arbitrary non-zero complex numbers and k_j are arbitrary integers. The involutive condition (I.144) for $u = 1$ now reduces to $\mathbf{U}(t)\tilde{\mathbf{U}}(t^{-1})^{-1} = \eta t^k \mathbf{1}$, where η is an arbitrary non-zero complex number and k is an arbitrary integer, which implies that

$$\eta_{l+1} = \pm 1 \tag{105}$$

and

$$\eta_{l-j} = \eta_{j+2}\eta_{l+1} \qquad k_{l-j} = k_{l+1} - k_{j+2} \tag{106}$$

for $j = 0,1,\ldots,\frac{1}{2}(l-3)$. Then $\eta_2,\eta_3,\ldots,\eta_{(l+1)/2}$ can be taken to be arbitrary non-zero complex numbers and $k_2,k_3,\ldots,k_{(l+1)/2}$ and k_{l+1} can be taken to be arbitrary integers.

Define $\mathbf{S}(t)$ to be the $(l+1)\times(l+1)$ diagonal matrix whose diagonal entries are given by

$$S(t)_{jj} = \begin{cases} \eta_{(l+1)/2}t^{k_{(l+1)/2}+\kappa} & \text{for } j = 1 \\ (\eta_{(l+1)/2}/\eta_j)t^{k_{(l+1)/2}-k_j+\kappa} & \text{for } j = 2,\ldots,(l-1)/2. \\ t^{\kappa} & \text{for } j = (l+1)/2 \\ 1 & \text{for } j = (l+3)/2,\ldots,l \end{cases}$$

where κ is an integer. Then

$$\mathbf{S}(t)\mathbf{U}(t)\tilde{\mathbf{S}}(t^{-1}) = \eta_{(l+1)/2}t^{k_{(l+1)/2}+\kappa}\begin{pmatrix} \mathbf{0} & \mathbf{K}_{(l+1)/2} \\ \eta_{l+1}t^{k_{l+1}-2\kappa}\mathbf{K}_{(l+1)/2} & \mathbf{0} \end{pmatrix}$$

where $\mathsf{K}_{(l+1)/2}$ is defined in (35). Thus, by the conjugacy condition (I.182), every type 2b involutive automorphism corresponding to set (F) is conjugate to one with $u = 1$ and either

$$\mathsf{U}(t) = \begin{pmatrix} 0 & \mathsf{K}_{(l+1)/2} \\ \eta_{l+1}\mathsf{K}_{(l+1)/2} & 0 \end{pmatrix}$$

or

$$\mathsf{U}(t) = \begin{pmatrix} 0 & \mathsf{K}_{(l+1)/2} \\ \eta_{l+1}t\mathsf{K}_{(l+1)/2} & 0 \end{pmatrix}$$

where η_{l+1} satisfies (105). However the type 2b involutive automorphisms with $u = 1$ corresponding to

$$\mathsf{U}(t) = \begin{pmatrix} 0 & \mathsf{K}_{(l+1)/2} \\ t\mathsf{K}_{(l+1)/2} & 0 \end{pmatrix}$$

and

$$\mathsf{U}(t) = \begin{pmatrix} 0 & \mathsf{K}_{(l+1)/2} \\ -t\mathsf{K}_{(l+1)/2} & 0 \end{pmatrix}$$

are conjugate (by (I.182) with $s = -1$ and $\mathsf{S}(t) = \mathbf{1}_{l+1}$). Moreover the type 2b involutive automorphism with $u = 1$ corresponding to $\mathsf{U}(t) = \mathsf{K}_{l+1}$ is conjugate to the Cartan involution, for with $\mathsf{S}(t)$ defined by (64), then

$$\mathsf{S}(t)\mathsf{K}_{l+1}\tilde{\mathsf{S}}(t^{-1}) = 2\begin{pmatrix} \mathbf{1}_{(l+1)/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{(l+1)/2} \end{pmatrix}$$

where the matrix on the right-hand side here is merely a rearrangement of a special case of (17), and hence, as shown in subsection 7.1, gives an automorphism that is conjugate to that corresponding to $\mathsf{U}(t) = \mathbf{1}_{l+1}$.

Thus there are *only two new* conjugacy classes of type 2b involutive automorphisms with $u = 1$. Their representatives may be taken to be

(1) the involutive automorphism corresponding to

$$\mathsf{U}(t) = \begin{pmatrix} 0 & \mathsf{K}_{(l+1)/2} \\ -\mathsf{K}_{(l+1)/2} & 0 \end{pmatrix} \tag{107}$$

which (by (I.70), (I.71), (I.73)) and (I.145)) is

$$\begin{aligned}
&\psi(h_{\alpha_0}) = -h_{\alpha_0} - 2\sum_{j=1}^{l} h_{\alpha_j} \\
&\psi(h_{\alpha_j}) = h_{\alpha_{l+1-j}} \qquad (j = 1, \ldots, l) \\
&\psi(c) = -c \\
&\psi(d) = -d \\
&\psi(e_{\pm\alpha_0}) = -e_{\mp\{\alpha_0+2\alpha_1+\ldots+2\alpha_l\}} \\
&\psi(e_{\pm\alpha_j}) = e_{\pm\alpha_{l+1-j}} \qquad (j = 1, \ldots, \tfrac{1}{2}(l-1), \tfrac{1}{2}(l+3), \ldots, l) \\
&\psi(e_{\pm\alpha_j}) = -e_{\pm\alpha_{l+1-j}} \qquad (j = \tfrac{1}{2}(l+1))
\end{aligned} \tag{108}$$

(2) the involutive automorphism corresponding to

$$\mathbf{U}(t) = \begin{pmatrix} 0 & \mathbf{K}_{(l+1)/2} \\ t\mathbf{K}_{(l+1)/2} & 0 \end{pmatrix} \tag{109}$$

which (by (I.70), (I.71), (I.73)) and (I.145)) is

$$\begin{aligned}
&\psi(h_{\alpha_j}) = -\sum_{j=1}^{l} h_{\alpha_j} \qquad (j=0)\\
&\psi(h_{\alpha_j}) = \sum_{j=1}^{l-j} h_{\alpha_j} + 2h_{\alpha_{l+1-j}} + \sum_{j=l+2-j}^{l} h_{\alpha_j} \qquad (j=1,\ldots,\tfrac{1}{2}(l-1))\\
&\psi(h_{\alpha_j}) = h_{\alpha_{l+1-j}} \qquad (j=\tfrac{1}{2}(l+1))\\
&\psi(h_{\alpha_j}) = \sum_{j=1}^{l-j} h_{\alpha_j} + \sum_{j=l+2-j}^{l} h_{\alpha_j} \qquad (j=\tfrac{1}{2}(l+3),\ldots,l)\\
&\psi(c) = -c\\
&\psi(d) = -(l+1)\{\sum_{j=1}^{(l+1)/2} jh_{\alpha_j} + \sum_{j=1}^{(l-1)/2} jh_{\alpha_{l-j+1}}\} - d\\
&\psi(e_{\pm\alpha_j}) = -e_{\mp\{\alpha_1+\ldots+\alpha_l\}} \qquad (j=0)\\
&\psi(e_{\pm\alpha_j}) = -e_{\pm\{\alpha_0+\ldots+\alpha_{l-j}+2\alpha_{l+1-j}+\alpha_{l+2-j}+\ldots+\alpha_l\}} \qquad (j=1,\ldots,\tfrac{1}{2}(l-1))\\
&\psi(e_{\pm\alpha_j}) = -e_{\pm\alpha_{l+1-j}} \qquad (j=\tfrac{1}{2}(l+1))\\
&\psi(e_{\pm\alpha_j}) = -e_{\pm\{\alpha_0+\ldots+\alpha_{l-j}+\alpha_{l+2-j}+\ldots+\alpha_l\}} \qquad (j=\tfrac{1}{2}(l+3),\ldots,l).
\end{aligned} \tag{110}$$

As the matrix $\mathbf{U}(t)$ of (107) is antisymmetric, and as that of (109) is symmetric for $t = 1$ and antisymmetric for $t = -1$, and as $\mathbf{K}_{l+1}$ is symmetric, the conjugacy conditions (I.182), (I.184), (I.186) and (I.188) all imply that the corresponding type 2b involutive automorphisms cannot be conjugate.

7.5. Summary of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b

The above considerations show that the number of conjugacy classes of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b is 1 if l is even and is 3 if l is odd, and that their representatives may be taken to be

(i) for l even or odd, the Cartan involution (103) which corresponds to $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = 1$;

(ii) for l odd, the automorphism (108), which corresponds to $u = 1$ and the matrix $\mathbf{U}(t)$ of (107);

(iii) for l odd, the automorphism (110), which corresponds to $u = 1$ and the matrix $\mathbf{U}(t)$ of (109).

8. Conclusions regarding the matrix formulation of the involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$

The analysis that has been given in the previous sections shows that the number of conjugacy classes of involutive automorphisms of $A_l^{(1)}$ for $l \geqslant 3$ of type 2b is $\frac{1}{8}(l^2+10l+48)$ if l is even and is $\frac{1}{8}(l^2+12l+99)$ if l is odd. These are distributed in the following.

(i) The number of conjugacy classes of *type 1a* involutive automorphisms with $u=1$ is $\frac{1}{2}(l+2)$ if l is even and is $\frac{1}{2}(l+5)$ if l is odd. Their representatives may be taken to be: (a) the identity automorphism; (b) the $[\frac{1}{2}(l+1)]$ automorphisms (23), which correspond to the matrices $\mathbf{U}(t)$ of (18), where the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (22); (c) for l odd, the involutive automorphism (39), which corresponds to the matrix of (18).

(ii) There is one conjugacy class of *type 1a* involutive automorphisms with $u=-1$. Its representative may be taken to be the automorphism (44), which corresponds to the matrix $\mathbf{U}(t)$ of (18), where the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (43).

(iii) The number of conjugacy classes of *type 1b* involutive automorphisms with $u=1$ is one if l is even and is three if l is odd. Their representatives may be taken to be: (a) for l even or odd, the involutive automorphism (55), which corresponds to the matrix $\mathbf{U}(t) = \mathbf{1}_{l+1}$; (b) for l odd, the involutive automorphism (56), which corresponds to the matrix $\mathbf{U}(t)$ of (54); (c) for l odd, the involutive automorphism (66), which corresponds to the matrix $\mathbf{U}(t)$ of (65).

(iv) There is one conjugacy class of *type 1b* involutive automorphisms with $u=-1$. Its representative may be taken to be the involutive automorphism (67), which corresponds to the matrix $\mathbf{U}(t) = \mathbf{1}_{l+1}$.

(v) The number of conjugacy classes of *type 2a* involutive automorphisms is $\frac{1}{8}(l+2)(l+4)$ if l is even and is $\frac{1}{8}(l+3)(l+5)$ if l is odd. Their representatives may be taken to be: (a) the automorphism (73) which corresponds to $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u=1$; (b) the $l - [\frac{1}{2}l]$ automorphisms (76), which correspond to $u=1$ and the matrices $\mathbf{U}(t)$ of (17), where $k_j = 0$ for every $j = 2, 3, \ldots, l$ and the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (74); (c) the $[\frac{1}{2}(l+1)]$ automorphisms (93), which correspond to $u=1$ and the matrices $\mathbf{U}(t)$ of (88), where N_+ and N_t satisfy (89) and (90); (d) the automorphisms (96), which correspond to $u=1$ and the matrices $\mathbf{U}(t)$ of (78) with N_+, N_-, and N_t satisfying (79), (80), (81) and (82), (the number of conjugacy classes of this kind being $\frac{1}{8}l(l-2)$ if l is even and $\frac{1}{8}(l-1)(l+1)$ if l is odd).

(vi) The number of conjugacy classes of *type 2b* involutive automorphisms is one if l is even and three if l is odd. Their representatives may be taken to be: (a) for l even or odd, the Cartan involution (103) which corresponds to $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u=1$; (b) for l odd, the automorphism (108), which corresponds to $u=1$ and the matrix $\mathbf{U}(t)$ of (107); (c) for l odd, the automorphism (110), which corresponds to $u=1$ and the matrix $\mathbf{U}(t)$ of (109).

These results may be compared with those obtained by Kobayashi [4] for the derived algebra of $A_l^{(1)}$ for $l \geqslant 3$ by another method. (As mentioned previously, as Kobayashi considered *only* the *derived* algebra of $A_l^{(1)}$, his analysis did not include any discussion of the action of automorphisms on the scaling element d.) In Kobayashi's classification the order 2 automorphism conjugacy class representatives are called (a), (a'), (b'), (b''), (b'''), (c), (d), (d'), (d''), (e), (e'), (e'') and (f). The extensions of these to the whole of the Kac–Moody algebra $A_l^{(1)}$ will be now related to the conjugacy classes of $A_l^{(1)}$ listed at the beginning of this section. Unfortunately the information given by Kobayashi [4] is not sufficiently explicit to make an absolutely firm identification in every case, as the following list indicates:

(a) The extensions ψ_a of Kobayashi's involutive automorphisms (a) are the type 1a automorphisms (23), which correspond to $u = 1$ and the matrices $\mathbf{U}(t)$ of (18), where the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (22).

(a') (for l odd)]. The extension $\psi_{a'}$ of Kobayashi's involutive automorphism (a') is probably conjugate to the type 1a involutive automorphism (39), which corresponds to $u = 1$ and the matrix of (18);

(b) The extensions ψ_b of Kobayashi's involutive automorphisms (b) are the type 2a involutive automorphisms corresponding to $u = 1$ and

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{N_+-1}, \underbrace{-t, \ldots, -t}_{N_t}, \underbrace{-1, \ldots, -1}_{l+1-N_+-N_t}, \underbrace{t^2}_{1}\}$$

and so are conjugate to the type 2a involutive automorphisms which correspond to $u = 1$ and the matrices $\mathbf{U}(t)$ of (78) with N_+, N_-, and N_t satisfying (79), (80), (81) and (82).

(b') The extensions $\psi_{b'}$ of Kobayashi's involutive automorphisms (b') are the type 2a involutive automorphisms corresponding to $u = 1$ and

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{N_+-1}, \underbrace{-t, \ldots, -t}_{l+1-N_+}, \underbrace{t^2}_{1}\}$$

and so are conjugate to the type 2a involutive automorphisms (93), which correspond to $u = 1$ and the matrices $\mathbf{U}(t)$ of (88), where N_+ and N_t satisfy (89) and (90).

(b'') The extensions $\psi_{b''}$ of Kobayashi's involutive automorphisms (b'') are the type 2a involutive automorphisms corresponding to $u = 1$ and

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{N_+-1}, \underbrace{-1, \ldots, -1}_{l+1-N_+}, \underbrace{t^2}_{1}\}$$

and so are conjugate to the type 2a involutive automorphisms (76), which correspond to $u = 1$ and the matrices $\mathbf{U}(t)$ of (17), where $k_j = 0$ for every $j = 2, 3, \ldots, l$ and the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (74).

(b''') The extension $\psi_{b'''}$ of Kobayashi's involutive automorphism (b''') is the type 2a involutive automorphism corresponding to $u = 1$ and

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{l}, \underbrace{t^2}_{1}\}$$

and so is conjugate to the type 2a involutive automorphism (73), which corresponds to $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = 1$.

(c) The extension ψ_c of Kobayashi's involutive automorphism (c) is the type 1a automorphism (44), which corresponds to $u = -1$ and the matrix $\mathbf{U}(t)$ of (18), where the $\{\eta_2, \eta_3, \ldots, \eta_{l+1}\}$ satisfy (43);

(d) The extension ψ_d of Kobayashi's involutive automorphism (d) is the type 1b involutive automorphism corresponding to $u = 1$ and

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{l}, \underbrace{t^2}_{1}\}$$

and so is conjugate to the type 1b involutive automorphism (55), which corresponds to $u = 1$ and the matrix $\mathbf{U}(t) = \mathbf{1}_{l+1}$.

(d') The extension $\psi_{d'}$ of Kobayashi's involutive automorphism (d') is the type 1b involutive automorphism corresponding to $u = 1$ and

$$\mathbf{U}(t) = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{l-1}, \underbrace{-t}_{1}, \underbrace{t^2}_{1}\}$$

and so is conjugate to the type 1b involutive automorphism (56), which corresponds to $u = 1$ and the matrix $\mathbf{U}(t)$ of (54).

(d'') (for l odd). The extension $\psi_{d''}$ of Kobayashi's involutive automorphism (d'') is probably conjugate to the type 1b involutive automorphism (66), which corresponds to $u = 1$ and the matrix of (65).

(e) The extension ψ_e of Kobayashi's involutive automorphism (e) is the Cartan involution (103), which is a type 2b automorphism that corresponds to $\mathbf{U}(t) = \mathbf{1}_{l+1}$ and $u = 1$;

(e) and (e') (for l odd). The expressions quoted by Kobayashi [4] are not sufficently explicit to allow the general form of the automorphisms to be deduced but their extensions are probably conjugate to the type 2b involutive automorphisms (108) or (110), which correspond to $u = 1$ and the matrices of (107) and (109).

(f) The extension ψ_f of Kobayashi's involutive automorphism (f) is the type 1b automorphism (67), which corresponds to $u = -1$ and to the matrix $\mathbf{U}(t) = \mathbf{1}_{l+1}$.

As shown in section 4.2, the two involutive automorphisms (d) and (d') of Kobayashi lie in the *same* conjugacy class if l is *even*, but lie in *different* conjugacy classes if l is *odd*. This conclusion differs from that of Kobayashi [4], who implies that (d) and (d') *always* lie in *different* conjugacy classes. (This matter was analysed in detail for the case $l = 2$ in section 8 of paper III.)

The involutive automorphisms of $A_l^{(1)}$ have also been considered previously by Levstein [5], but unfortunately his tables are insufficiently explicit to allow a detailed comparison to be made with the results obtained here.

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